INTRODUCTION

TO

DYNAMICS

BY

M. RAY, D. Sc.

Professor and Head of the Department of Mathematics,

Agra College, Agra.

AND

S. D. NIGAM, M. Sc., Ph. D.

Department of Mathematics,

Agra College, Agra.

S. CHAND & CO.

PUBLISHERS

FOUNTAIN - - DELHI

Published by :
G. S. Sharma, Manager
S. Chand & Co.
Fountain, Delhi.

Printed at :
Oxford & Cambridge Press
Urdu Bazar Delhi.

PREFACE

This book is an introduction to the subject of Dynamics and is intended to serve as a first course on the subject. Its purpose is, frankly, pedagogical. Attempt has been made to present the subject matter in as clear and lucid a manner as possible. All important principles have been explained in their historical setting and have been illustrated by means of solved examples. Some important dynamical concepts, like those of force, mass and rectilinear motion have been examined critically and placed in a proper perspective. Numerous examples have been selected from various standard texts on the subject and Examination Papers. The book is complete as far as it goes and it is hopced that nothing of importance has been left out. It has been so written as to meet the requirements of the students appearing at the Intermediate Engineering and other competitive examinations.

If the book proves of any help to those for whom it is intended, the authors will consider their labours amply rewarded.

Any suggestions for improvement or corrections of errors will be thankfully received.

Agra College,
AGRA
May, 1951

M. RAY
S. D. NIGAM

CONTENTS

DYNAMICS

CHAPTER I

INTRODUCTORY

1. All **Physical Sciences** of which **Mechanics** is a branch depend for the description of phenomena on three fundamental concepts : **Space, Time** and **Matter**.

Space. 'Absolute space, in its own nature and without regard to anything external, always remains similar and immovable.'

Relative space is some movable dimension or measure of absolute spaee, which our senses determine by its position with respect to other bodies, and which is commonly taken for immovable absolute space.

The unit of space is length.

Time. 'Absolute, true, and Mathematical time, of itself and by its own nature, flows uniformly on, without regard to anything external. It is called **duration**.'

'Relative, apparent, and common time, is some sensible and external measure of absolute time, estimated by the motion of bodies.' *e. g.* an hour, a day, a month, a year.

Matter. Matter is that which occupies space and can be perceived by the senses, *e. g.* wall, iron, water, oxygen. Its fundamental property is **Inertia** (tendency to resist motion).

2. **Measurement of Space, Time and Matter.** We all know these three fundamental quantities intuitively. But they can be measured also with sufficient accuracy. *The measure of space is length, that of time is duration and that of matter is mass.*

Mass is the quantity of matter contained in a body.

Body is the limited quantity of matter occupying some definite space, *e. g.* a piece of wood.

A **Rigid body** is a body different parts of which always occupy the same relative position, *e. g.* a piece of marble. Liquids and gases are not rigid bodies.

A **Particle** is a portion of matter indefinitely small in dimensions that the distances between the different portions of it may be neglected for all practical purposes. It may be supposed to be identical with a geometrical point, having position only but no magnitude. Sometimes bodies of finite size are treated as particles.

3. Fundamental Units. F. P.S. System. The system of units commonly used in England, in which the fundamental units are *a foot, a second* and *a pound,* is called the Foot-Pound Second system.

C. G. S. System. The fundamental units are *a centimetre, a gramme*, and *a second.* This is called the **Metric System** also.

4. Rest and Motion. A body is said to be at rest if it does not change its position *relative*to the surrounding objects, (though it might be actually moving in space).

A body is said to be in motion, if its position relative to the surrounding objects is changing.

Note. There is no absolute rest or absolute motion in nature. **Rest** and **Motion** in themselves are meaningless concepts. We have no criteria to know them. Even the hypotheses of absolute space and absolute time are redundant. They are abstractions in thought, metaphysical in their character and have no place in the Physical Sciences. All that we know about matter and material bodies is the result of measurement. But measurement is always *relative and depends on the conditions under which the measurements are performed.* Hence the data, with which every science starts, does not give us any justification for believing in absolute space, absolute time, absolute rest and absolute motion.

5. Motion of a Particle. A particle has only one kind of motion, called the motion of translation. It only changes its position ; it cannot rotate about itself. It can move either along a straight line or along a curved path. Its motion can be uniform or variable.

6. Uniform and Non-uniform (variable) Motion. If a particle describes equal distances of homogeneous space in equal intervals of time, however short the intervals may be, its motion is called **uniform motion**.

If the particle describes unequal distances of homogeneous space in equal intervals of time, however short the intervals may be, its motion is called **non-uniform or variable motion.**

7. Types of Motion.

(*i*) Uniform motion in a straight line. } **Rectilinear motion**
(*ii*) Variable motion in a straight line. }

(*iii*) Uniform motion along a curved path.

(*iv*) Variable motion along a curved path.

8. Some Remarks on the Cause of Motion. Let us analyse human experience. A ball is placed on the ground and we say, it is at rest relative to the earth. Mr. Shaw *hits* the ball with a hockey stick and the ball, if the *hit is sufficient*, begins to move relative to the earth and after some time comes to rest again. There is surely some difference between the state of the ball before and after the hit. One simple explanation of the change may be given in the following manner :—

Newton has said, "Every body continues in its state of rest, or of uniform motion in a straight line except in so far as it be compelled by *external impressed force* to change that state."

Comparing the example cited above with Newton's statement it is at once obvious that '*hit*' is synonymous with the *external impressed force.* Experience shows that if the hit is not sufficiently intense, the ball may not move at all ; but if the intensity of the hit is gradually increased, there will come a time when the ball begins to move. Hence the hit may be looked upon as something which tends to change the state of rest of the ball.

What happens to the ball if we imagine that there is no resistance to its motion ? The second part of Newton's statement at once comes to our aid, that the ball will continue moving uniformly in a straight line. Here also experience

shows that if the ground be made more smooth the ball moves for a greater length of time. Therefore it is quite reasonable to infer that if the resistance were completely removed the ball would move for ever. But there is one limitation in this concept of the cause of motion.

So long as the forces, conceived as the cause of motion, are confined to *thrusts* and *tensions*, where there is supposed to be direct contact of material, the hypothesis works quite well. But it cannot be applied to cases of *Action at a distance*, that is, where there is no material contact between the cause of motion and the body which moves under the influence of that cause. The action of sun on earth cannot be explained on this hypothesis. Hence force must be looked upon, not as something given but as a convenient hypothesis which describes certain motion of the material bodies.

9. A Classification. All quantities may be divided into two classes : **Vectors** and **Scalars**.

A quantity which has magnitude, direction and sense is called a **Vector** quantity or simply a **Vector**. Vectors are also called **directed quantities**, as they can be represented in magnitude, direction and sense by means of straight lines of definite length, *e.g.* force, displacement, velocity, acceleration.

A quantity which has magnitude only is called a **Scalar** quantity. *e.g.* mass, time, density.

10. Mechanics Defined. The name Mechanics was originally used to designate the science of making machines. It is now applied to the whole theory which treats of bodies in equilibrium, as well as in motion, with or without reference to mass or force.

It has two branches : Statics and Dynamics.

Statics is the science that treats of the action of forces on bodies at rest ; or more rightly, of the forces in equilibrium.

Dynamics is the science which treats of bodies in motion.

CHAPTER II

RECTILINEAR MOTION

11. When a particle moves in such a way that its path obtained by joining its positions at different instants is a straight line, the motion is called **rectilinear motion**. The motion may or may not be uniform.

12. Can a line be straight : We do not deny that it is not possible to think of a line as straight. We simply enquire if there is anything in nature which may be called a straight line in the strictest sense of the term ; in other words, if there is any physical basis for believing in this concept ? The idea of a straight line is introduced in the earlier classes during the study of Euclid's Geometry. A straight line is defined as the shortest distance joining two points. But how do we know that of a number of distances between two points there is one and only one which is the shortest ? The answer at once suggests itself that by measurement with a suitable scale we can at once decide the question. But what does this suitable scale mean ? The word suitable here implies, that we shall take a scale which is itself straight and not a curved one. Hence it is clear that to judge the straightness of a line we require a scale which is straight. But how do we know that the scale which we have chosen is really a straight one ? The only answer is that we shall test it by means of another scale, and this testing will go on for ever because we never know whether the first scale with which we started was really a straight one. Some of our enthusiastic readers may say that *light travels in straight lines* and that by using delicate optical instruments one can easily by comparison decide the question of a material scale being perfectly straight. But recent researches in Physics have established it beyond the shadow of doubt that the track of a light ray is slightly curved and that it is not perfectly straight. This settles the question that in the present state of our knowledge there is nothing in nature which corresponds to our idea of a straight line. This, however, does not in any way lessen the importance of the straight line or of the

rectilinear motion, because in the world of every day experience we have to deal with cases which for all practical purposes may be considered as those of rectilinear motion. Therefore a science based upon the concept of straight lines cannot be of a universal character.

13. If at any instant the position of a moving point be P, and at any subsequent instant it be Q, then PQ is the change in its position in the intervening time.

Fig. 1

14. Speed. *The speed of a moving point is the rate at which it describes its path, without any reference to direction.*

The speed is uniform or constant when the point moves through equal lengths of its path in equal intervals of time, however small these intervals may be. A ship sailing at the rate of 60 miles an hour cannot be said to be sailing at a constant rate, unless it sails one mile in one minute, 88 feet in one second, 1 foot in $\frac{1}{88}$ seconds, one-millionth of 60 miles in each one-millionth of an hour and so on.

When uniform, the speed of a point is measured by the distance traversed by it in a unit of time. At every instant of its motion the speed has a constant value.

When non-uniform or variable the speed changes from instant to instant and at no two different instants the speed has the same value. Hence to describe motion it is necessary to mention the instant at which the speed is given and also to give the manner or law according to which the speed changes either with respect to time or with respect to distance.

The variable speed may be measured at any instant in the following way : In the figure of Art. 13, if PQ$=s$ be the distance traversed in one second, and s_1, s_2, s_3......be the distances traversed in $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$,......second rsespectively, then in the infinite collection $\left(s, \frac{s_1}{\frac{1}{10}}, \frac{s_2}{\frac{1}{100}}, \frac{s_3}{\frac{1}{1000}}, \ldots\ldots\ldots \right)$ each ratio gives a nearer approximation to the speed at P than all the preceding ratios.

In the language of mathematics this can be put as follows : *Let s be the length of the portion of the path described by the moving point in the small time t following the instant under consideration, then the ultimate value of s/t as the time t is taken smaller and smaller, is the measure of the speed of the moving point at the instant under consideration.*

15. Displacement. The displacement of a moving point is its change of position. In the figure of Art. 13, if the particle is moving from P to Q in the direction PQ, the displacement is PQ. Hence the displacement of a point involves both magnitude and direction. It is therefore a vector quantity.

16. Velocity. *The velocity of a moving point is the rate of its displacement.*

A velocity therefore possesses both magnitude and direction. It is therefore a vector quantity ; while the speed has reference to magnitude only and is therefore a scalar quantity.

The distinction between velocity and speed becomes more clear when the point is moving along a curved path.

A point is said to be moving with uniform or constant velocity, when it is moving in a fixed direction, and passes over equal lengths in equal times, however small these times may be.

When uniform, the velocity of a moving point is measured by its displacement per unit of time ; when variable, the method of measurement is same as in Art. 14.

17. If the velocity of a particle when moving from P to Q be denoted by v, then, if it be moving from Q to P with a velocity v, we can say,:that it has a velocity $-v$ in the direction PQ in the second case.

Hence a velocity 5 miles per hour due east, and -5 miles per hour due west are two different ways of stating the samo fact.

The expression 3 feetepecondsism nodr ply 3 ft./sec. means that the particle describes 3 feet in rone .seco

18. Average Velocity. If a particle moving either with a uniform velocity or with a non-uniform velocity along a straight line, describes a distance s in time t, then, $\frac{s}{t}$ is called its average velocity during the interval of time t.

If the motion is uniform. the average velocity and the uniform velocity are identical.

If the motion is non-uniform, the average velocity in a given interval of time is the velocity, with which if the particle moves uniformly, it will describe the same path as the given particle in the given time.

19. Relative Velocity. The relative velocity of a point B with respect to a point A is the rate of change of B's position relative to A, in a definite sense and direction.

Consider the case of two trains moving on parallel rails in the same direction with equal velocities. Let A and B be two points, one on each train. The line AB would remain constant in magnitude and direction, and the velocity of B relative to A would be zero.

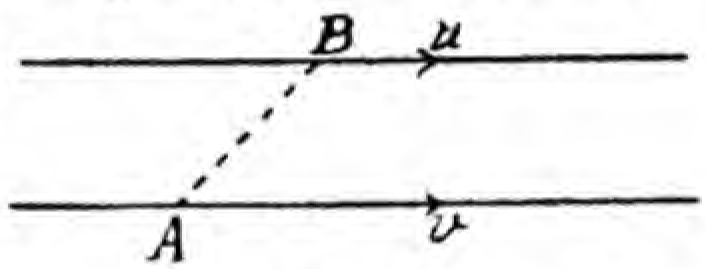


Fig. 2

Next, if one train moves at the rate of 25 miles per hour and the other at the rate of 20 miles per hour in the same direction, the distance AB will no more be a constant. If we neglect the distance between the parallel rails, then, the point B is gaining 5 miles every hour over A. Hence the relative velocity of B with respect to A is 5 miles per hour.

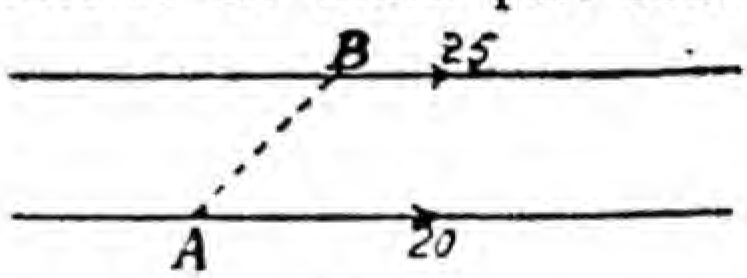


Fig. 3

Again, in the above illustration, if the two trains are moving in opposite directions, B is gaining at the rate of 45 miles per hour over A, which is therefore the velocity of B relative to A.

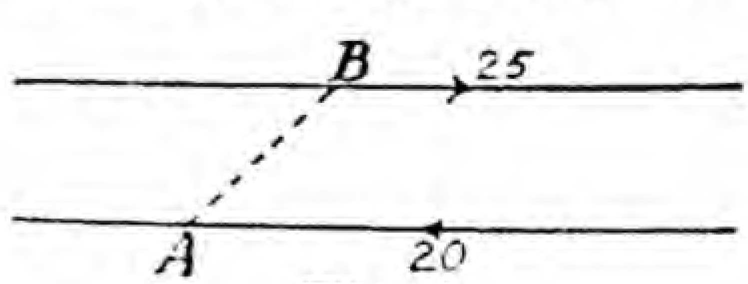


Fig. 4

20. Compounding of Velocities. Theorem of Parallelogram of Velocities. *If a moving point possesses simultaneously velocities which are represented in magnitude and direction by the two sides of a parallelogram drawn from a point, they are equivalent to a velocity which is represented in magnitude and direction by the diagonal of the parallelogram passing through the point.*

Consider an insect P crawling along a straight rod AB with a velocity u, while the rod itself is being moved in some other direction AD with a velocity v. Initially the insect is at A. At the end of $\frac{1}{2}$ second, the rod AB will be in the position A_1B_1 and the insect will be at P, where $AA_1=\frac{v}{2}$, $A_1P=\frac{u}{2}$. Again, at the end of one second the rod AB will be in the position DC, and the insect will be at C. Hence at each instant of motion the insect will be somewhere on the line AC, and since the two co-existent velocities are constant in magnitude and direction, the velocity from A to C is also constant both in magnitude and direction.

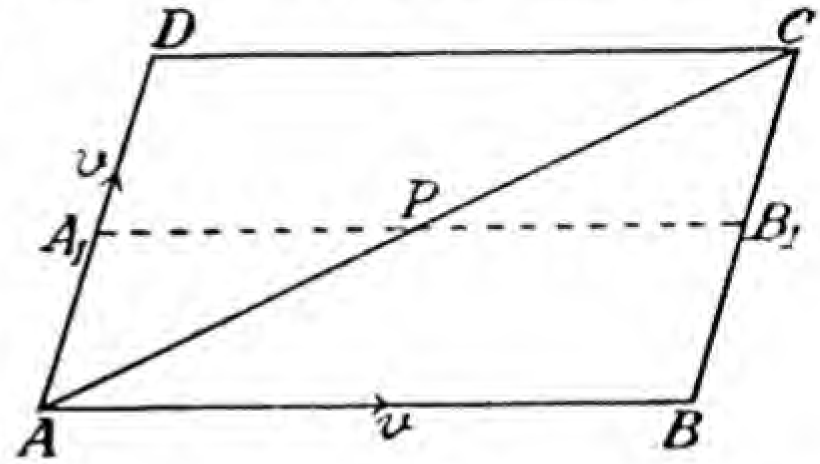


Fig. 5

Therefore AC represents a velocity which is equivalent to the velocities AB and AD, both in magnitude and direction. Whether the insect moves along AC with a velocity represented by $\overrightarrow{AC}$, or whether it possesses simultaneously two velocities represented by $\overrightarrow{AB}$ and $\overrightarrow{AD}$, the effect will be same in both cases. Starting from A it will reach C at the end of one second in both cases.

The velocity $\overrightarrow{AC}$ is called the **resultant** of the velocities $\overrightarrow{AB}$ and $\overrightarrow{AD}$, and in the vector notation, it is written as $\overrightarrow{AB}+\overrightarrow{AD}=\overrightarrow{AC}$.

It is to be noted that the left hand side does not mean a scalar sum ; it on the contrary represents a vector sum, which means that the velocities $\overrightarrow{AB}$ and $\overrightarrow{AD}$ have been compounded according to the law of parallelogram of velocities.

The velocities $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are called the **components** of the velocity $\overrightarrow{AC}$.

21. *To find the resultant of two velocities u and v inclined at an angle α to one another.* Let the velocities u and v be represented in magnitude and direction by the lines AB and AD, and let $\angle BAD = \alpha$. Complete the parallelogram with AB and AD as adjacent sides and let AC be the diagonal. Then, by the parallelogram law of velocities, AC will represent in magnitude and direction the resultant of the velocities u and v.

From C draw CE perpendicular to AB. Then,

$AC^2 = AE^2 + CE^2$
$= (AB + BE)^2 + CE^2$
$= (AB + BC.\cos\alpha)^2 + BC^2.\sin^2\alpha$
$= AB^2 + BC^2 + 2AB.BC.\cos\alpha$
$= AB^2 + AD^2 + 2AB.AD\cos\alpha$

If the resultant velocity be represented by w,

$$w^2 = u^2 + v^2 + 2uv\cos\alpha.$$

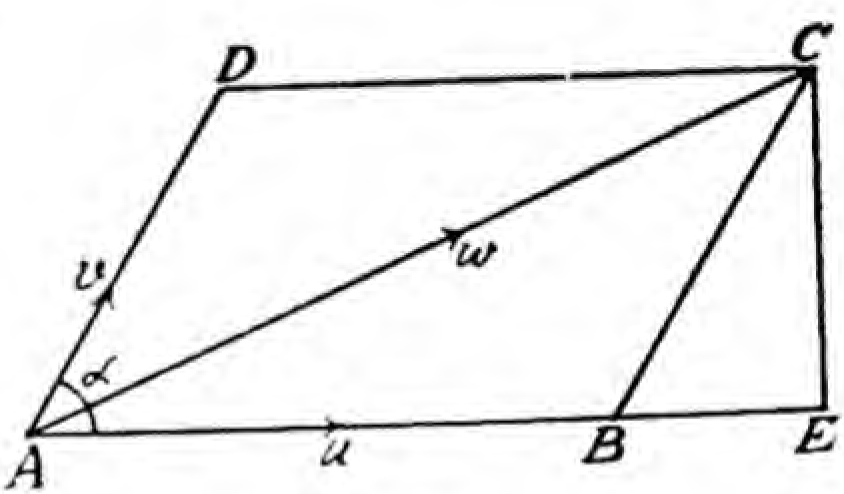


Fig. 6

The angle which the resultant makes with the direction of u is given by, $\angle CAB = \theta$.

$$\tan\theta = \frac{CE}{AE} = \frac{CE}{AB + BE}$$

$$= \frac{v\sin\alpha}{u + v\cos\alpha}.$$

Cor. I. If the velocities u and v be at right angles ; *i.e.*, if $\alpha = 90°$, then $w^2 = u^2 + v^2$ and $\tan\theta = \dfrac{v}{u}$.

Cor. II. If the velocities be equal ; *i.e.*, $u = v$

$$w^2 = 2u^2(1 + \cos\alpha)$$

$$=4u^2 \cos^2\frac{\alpha}{2}$$

$$w=2u \cos\frac{\alpha}{2}.$$

$$\text{and } \tan \theta=\frac{\sin \alpha}{1+\cos \alpha}=\frac{2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}}{2 \cos^2\frac{\alpha}{2}}=\tan\frac{\alpha}{2}.$$

$$\therefore \qquad \theta=\frac{\alpha}{2}.$$

Thus, in this case the resultant bisects the angle between the components.

22. A given velocity w represented in magnitude and direction by $\overrightarrow{AC}$ can be resolved into two component velocities in an infinite number of ways ; because an infinite number of parallelograms can be constructed with the given line AC as the diagonal. The sides AB and AD of one such parallelogram represent the two component velocities both in magnitude and direction.

In a particular case when the component velocities are perpendicular to one another, they are called the *resolved parts* of the velocity w. In this case the parallelogram degenerates into a rectangle.

The resolved parts of the velocity w in a given direction making an angle θ with the direction of AC and perpendicular to the given direction can be found, by simply constructing a rectangle, of which AC is the diagonal. Draw a line making an angle θ with AC through the point A and complete the rectangle ABCD. By the parallelogram law, AB and AD represent the resolved parts of w in the required directions, both in magnitude and direction *i.e.*, $w \cos\theta$ and $w \sin\theta$ respectively.

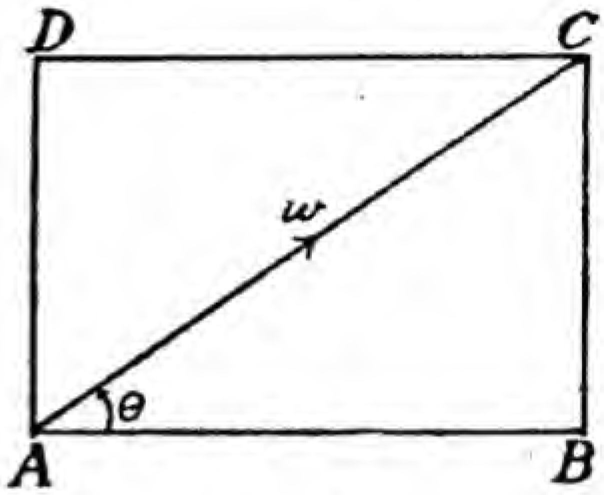


Fig. 7

In Fig. 8 the resolved parts of w $(\vec{AC})$ are, according to the above rule $\vec{AB'}$ and $\vec{AD}$. Hence the resolved parts in the directions AB and AD are $\vec{AB'}$ *reversed* and $\vec{AD}$.

$$\vec{AB'}=w\cos CAB'$$
$$=w\cos(\pi-\theta)$$
$$=-w\cos\theta.$$

$$\therefore\quad \vec{AB'}\text{ reversed}=-(\vec{AB'})$$
$$=w\cos\theta.$$

The resolved part along AD is $w\sin\theta$.

In either case the resolved part of w along AB is $w\cos\theta$.

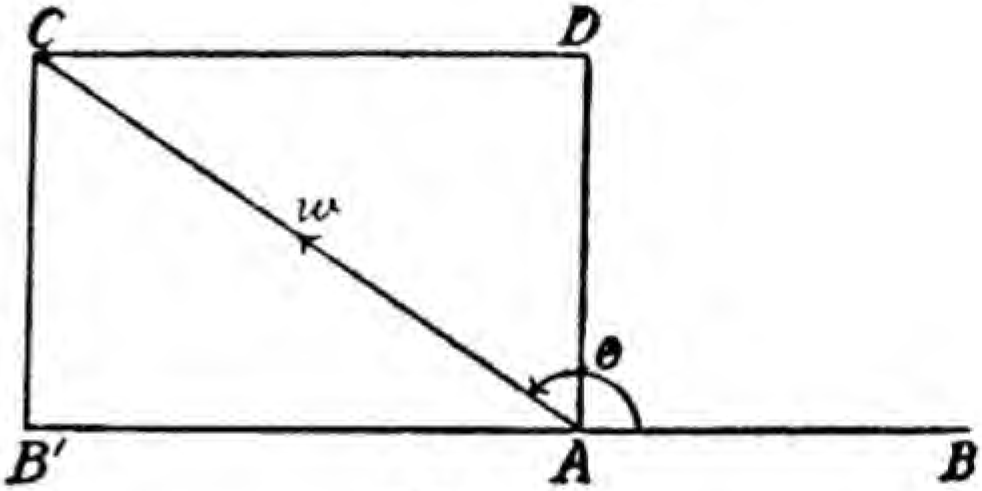


Fig. 8

Thus the resolved part of a given velocity in a given direction is equal to the product of the given velocity and the cosine of the angle which the given velocity makes with the given direction.

In either case the resolved part of w along AD is $w\sin\theta$.

Thus the resolved part of a velocity in a direction perpendicular to a given direction is equal to the product of the velocity and the sine of the angle which the given velocity makes with the given direction.

Cor. I. When $\theta=0$; *i.e.*, $\cos\theta=1$, the resolved part is w.

The resolved part of a velocity in its own direction is the original velocity itself.

Also, $\sin\theta=0$, the resolved in the direction perpendicular to the given direction becomes zero.

Hence a particle on account of its velocity in a particular direction only, cannot move perpendicular to that direction.

Ex. 1. Resolve 5 ft./sec. along OX and OY, given that it makes an angle of 30° with the direction OX.

Resolved part along $OX=5\cos 30^\circ$
$$=\frac{5\sqrt{3}}{2} \text{ ft./sec.}$$

Resolved part along $OY=5 \sin 30^\circ=\frac{5}{2}$ ft./sec.

Ex. 2. Resolve 15 cms./sec. along OX and OY, given that it makes an angle of 120° with OX.

Resolved part along $OX=15 \cos 120^\circ=-\frac{15}{2}$ cms./sec.

Resolved part along $OY=15 \sin 120^\circ=\frac{15\sqrt{3}}{2}$ cms./sec.

23. *To find the components of a velocity in two given directions.* We wish to find the components of $u=\overrightarrow{AC}$ in the directions AB and AD where the angles CAB and CAD are α and β respectively. Complete the parallelogram ABCD, with AC as its diagonal. The components are $\overrightarrow{AB}$ and $\overrightarrow{AD}$.

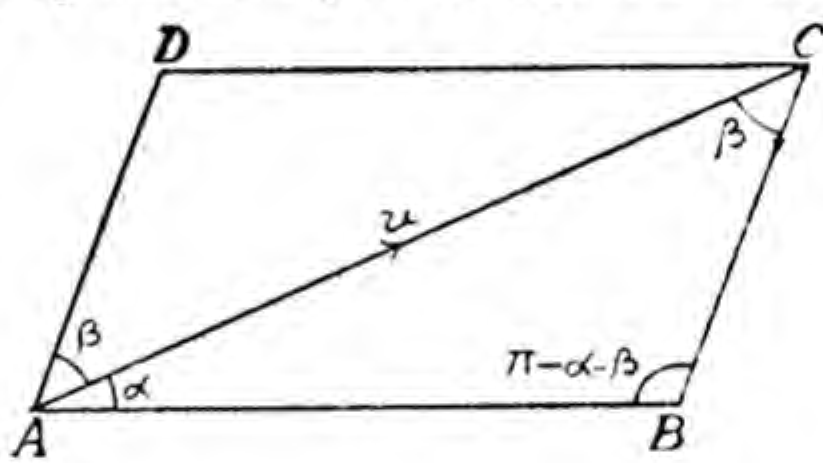


Fig.

In the triangle ABC,

$$\frac{AB}{\sin\beta}=\frac{BC}{\sin\alpha}=\frac{AC}{\sin(\pi-\alpha-\beta)}$$

$$\therefore \qquad \overrightarrow{AB}=u\frac{\sin\beta}{\sin(\alpha+\beta)}.$$

$$\overrightarrow{BC}=\overrightarrow{AD}=u.\frac{\sin\alpha}{\sin(\alpha+\beta)}.$$

If, $\alpha=\theta$ and $\alpha+\beta=\frac{\pi}{2}$, we have the case discussed in

Art. 22. Then,

$$\overrightarrow{AB} = u \sin\left(\frac{\pi}{2} - \theta\right) = u \cos \theta,$$

$$\overrightarrow{AD} = u \sin \theta.$$

24. Theorem of Triangle of Velocities. *If a particle possesses two simultaneous velocities represented in magnitude, direction and sense by two sides of a triangle, taken in order, the third side of the triangle, taken in the reverse order, represents their resultant in magnitude, direction and sense.*

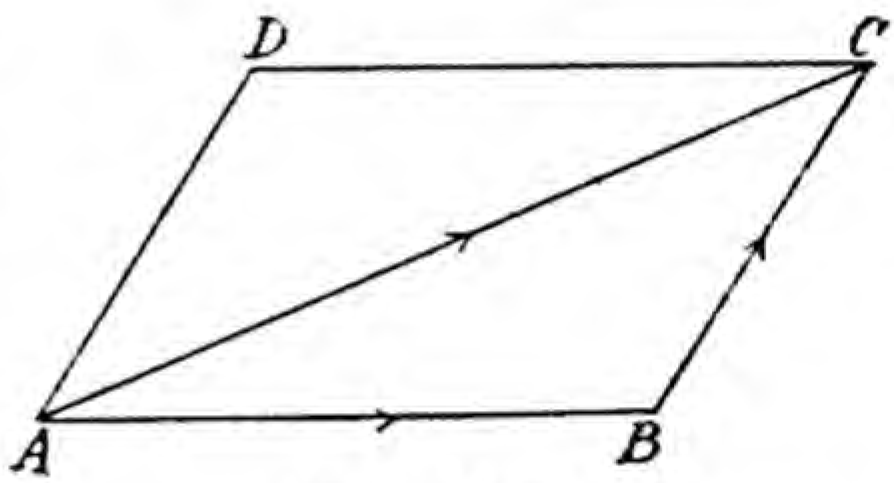


Fig. 10

The phrase "*taken in order*" means that if we start from A and travel round the triangle ABC, we should proceed from A to B, B to C, and C to A; we can also proceed from A to C, C to B, and B to A. In both these cases [AB, BC, CA] and [AC, CB, BA] the sides are said to be arranged in order. Clearly, the second order is the reverse of the first.

Hence if the two simultaneous velocities of the particle are $\overrightarrow{AB}$ and $\overrightarrow{BC}$, the resultant is $\overrightarrow{AC}$. This can at once be deduced from the law of parallelogram of velocites.

Complete the parallelogram ABCD.

The lines AB and BC represent the same velocities as the lines AB and AD. But AB and AD are the adjacent sides of a parallelogram, hence the diagonal AC represents their resultant, both in magnitude and direction.

In vector notation, we may express the above result as follows :—

We have to prove, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

L. H. S. $= \overrightarrow{AB} + \overrightarrow{AD}$ [Since AD = BC, being opposite sides of a parallelogram].

$= \overrightarrow{AC}$

Cor. 1. If a particle has simultaneously three velocities, which can be represented in magnitude and direction by the three sides of a triangle, taken in order, the particle will be at rest.

For, let $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$ be the three simultaneous velocities of the particle. The resultant of $\overrightarrow{AB}$ and $\overrightarrow{BC}$ is $\overrightarrow{AC}$ (By Art. 24). But $\overrightarrow{AC}$ and $\overrightarrow{CA}$ balance each other. Hence the particle is at rest.

25. *To find the resultant of two velocities* $\lambda.\overrightarrow{OA}$ and $\mu.\overrightarrow{OB}$ *along OA and OB respectively.*

Join AB and take a point C in AB, such that

$$\lambda.CA = \mu.CB \ldots\ldots \quad \ldots\ldots\ldots\ldots (1)$$

By Art. 24, the velocity $\lambda.\overrightarrow{OA}$ is equivalent to the velocities $\lambda.\overrightarrow{OC}$ and $\lambda.\overrightarrow{CA}$.

Also the velocity $\mu.\overrightarrow{OB}$ is equivalent to the velocities $\mu.\overrightarrow{OC}$ and $\mu.\overrightarrow{CB}$.

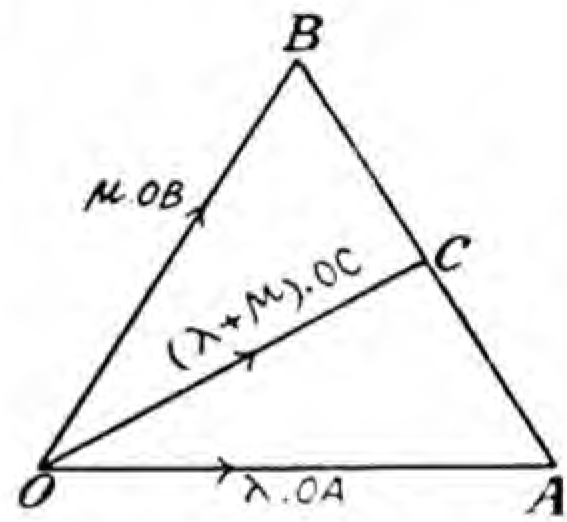


Fig. 11

The velocities $\mu.\overrightarrow{CB}$ and $\lambda.\overrightarrow{CA}$ destroy each other by virtue of (1), and the velocities $\lambda.\overrightarrow{OC}$ and $\mu.\overrightarrow{OC}$ are equivalent to the velocity $(\lambda+\mu).\overrightarrow{OC}$, which is therefore the resultant velocity.

26. Polygon of Velocities. If a particle possesses simultaneously velocities represented in magnitude and direction and sense by $(n-1)$ consecutive sides, taken in order, of a polygon of n sides, their resultant is represented by the remaining n^{th} side of the polygon taken in the reverse order. (The sides of the polygon need not be in one plane).

By Art. 24, the velocities $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are equivalent to $\overrightarrow{AC}$

Again the velocities $\overrightarrow{AC}$ and $\overrightarrow{CD}$ are equivalent to $\overrightarrow{AD}$. Continuing in this way, we have the last set of velocities $\overrightarrow{AF}$ and $\overrightarrow{FK}$, which are equivalent to the velocity $\overrightarrow{AK}$.

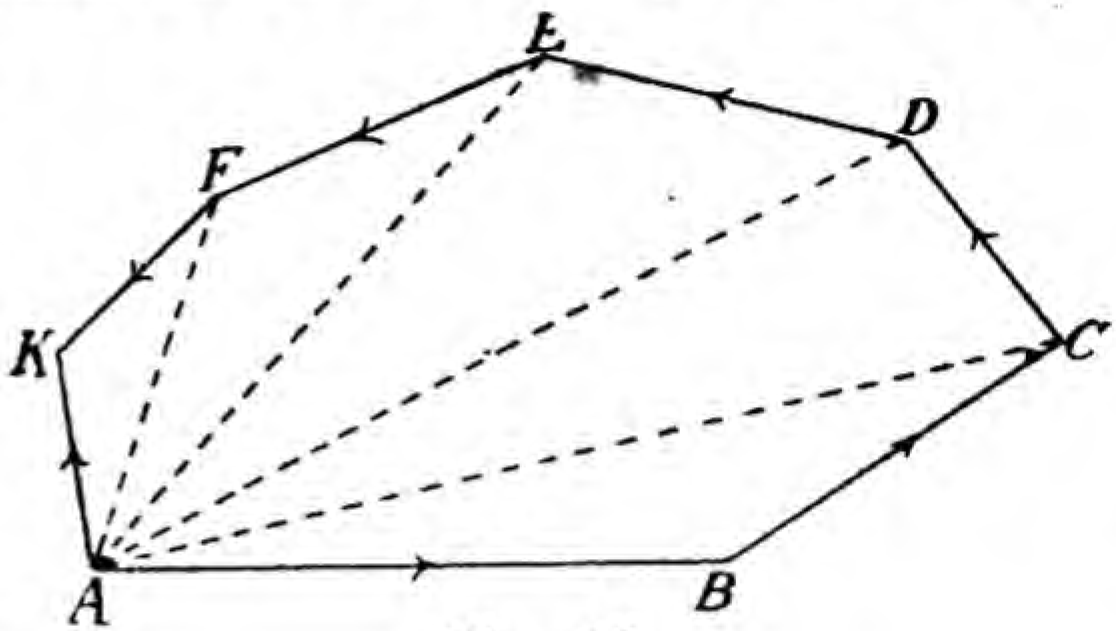


Fig. 12

27. *To find the resultant of any number of velocities of a particle in different directions in the same plane.*

Let $v_1, v_2, \ldots\ldots$ be the simultaneous velocities of a particle at O, all the velocities being in the same plane. Take any two perpendicular straight lines OX and OY and let $v_1, v_2, \ldots\ldots\ldots$ make angles $\theta_1, \theta_2 \ldots\ldots$ with OX respectively.

Resolve $v_1, v_2 \ldots\ldots$ along OX and OY. If V be the resultant velocity of O, making an angle θ with OX, then we must have,

$$V\cos\theta = v_1\cos\theta_1 + v_2\cos\theta_2 + \ldots = X \ldots\ldots(1)$$

$$V\sin\theta = v_1\sin\theta_1 + v_2\sin\theta_2 + \ldots = Y \ldots\ldots(2)$$

Squaring and adding

$$\begin{aligned} V^2 &= X^2 + Y^2 \\ &= (v_1\cos\theta_1 + v_2\cos\theta_2 + \ldots\ldots)^2 \\ &\quad + (v_1\sin\theta_1 + v_2\sin\theta_2 \ldots\ldots)^2 \\ &= v_1^2\cos^2\theta_1 + v_2^2\cos^2\theta_2 + \ldots\ldots \\ &\quad + 2v_1 v_2\cos\theta_1\cos\theta_2 + \ldots\ldots \\ &\quad + v_1^2\sin^2\theta_1 + v_2^2\sin^2\theta_2 + \ldots \\ &\quad + 2v_1 v_2\sin\theta_1\sin\theta_2 + \ldots\ldots \\ &= v_1^2 + v_2^2 + \ldots\ldots\ldots\ldots \\ &\quad + 2v_1v_2(\cos\theta_1\cos\theta_2 + \sin\theta_1 \\ &\qquad \sin\theta_2) + \ldots\ldots \end{aligned}$$

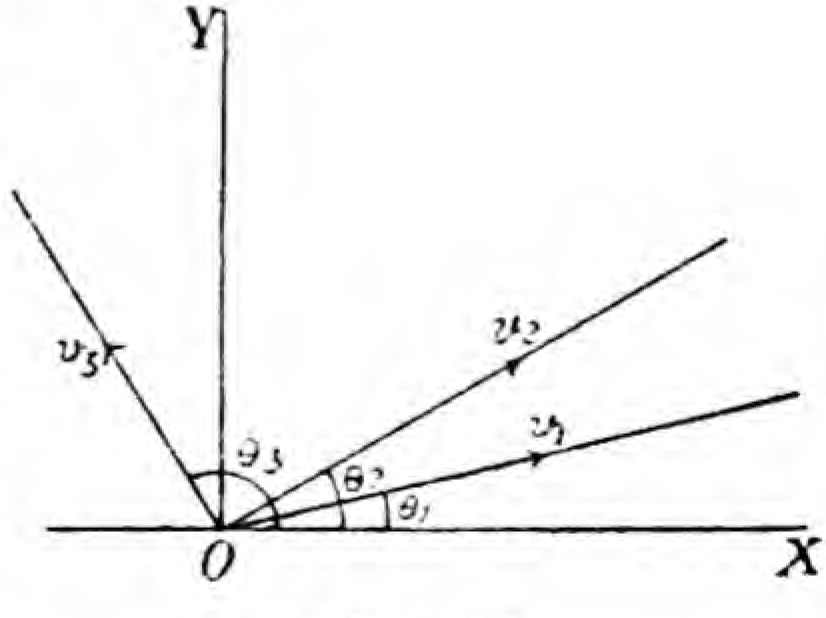


Fig. 13

$$=v_1^2+v_2^2+\ldots\ldots \qquad +2v_1v_2\cos(\theta_2-\theta_1)+\ldots\ldots$$

$$=v_1^2+v_2^2+\ldots\ldots \qquad +2v_1v_2\cos\widehat{v_1v_2}+2v_1v_3\cos\widehat{v_1v_3}+\ldots\ldots\ldots$$

where $\widehat{v_1v_2}$=angle between v_1 and v_2.

Dividing (2) by (1),

$$\tan\theta=\frac{Y}{X}=\frac{v_1\sin\theta_1+v_2\sin\theta_2+\ldots\ldots\ldots}{v_1\cos\theta_1+v_2\cos\theta_2+\ldots\ldots\ldots}.$$

Ex. 1. A truck is moving along a level road at the rate of 40 miles per hour. In what direction must a bullet be shot from it with a velocity of 352 yds./sec., so that its resultant motion be perpendicular to the truck.

The velocity of the truck

$$=\frac{40\times1760\times3}{60\times60}\text{ ft./sec.}$$

$$=\frac{176}{3}\text{ ft./sec.}$$

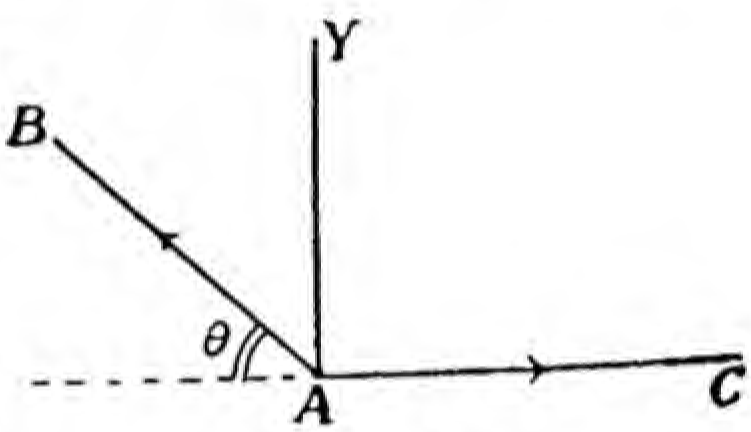


Fig. 14

Suppose the bullet is shot in the direction AB making an angle θ with CA produced. The velocity of the bullet is 1056 ft./sec. If the resultant motion of the bullet is along AY, then, the resolved part of the velocity of the bullet along CA must balance the velocity of the bullet due to the motion of the truck.

$$1056\cos\theta=\frac{176}{3}$$

$$\cos\theta=\frac{176}{3\times1056}=\frac{1}{18}$$

$$\text{or }\theta=\cos^{-1}\frac{1}{18}.$$

The velocity of the bullet along AY is

$$1056\sin\theta=\frac{1056\sqrt{323}}{18}=\frac{176}{3}\sqrt{323}\text{ ft./sec.}$$

Ex. 2. A mail train of length 100 yds. and a passengre train of length 150 yds. run on parallel lines at full speedr.

They pass each other in 5 secs., when they run in opposite directions, and in 30 secs., when they run in the same directions, Find the speed of each train in feet per second.

Let u be the speed of the mail train and v that of the passenger train in feet per second. In order to pass each other, each train has to travel 250 yds. or 750 feet.

The relative velocity of the mail train, when the trains are moving in opposite directions is $(u+v)$ ft./sec.

Therefore $\quad (u+v)\,5=750.$

or $\quad u+v \quad =150.$

Again, when the trains are moving in the same direction, the relative velocity of the mail train is $(u-v)$ ft./sec.

Therefore $\quad (u-v)\,30=750$

or $\quad u-v \quad =25.$

Therefore $\quad u=\dfrac{175}{2}$ ft./sec., $\quad v=\dfrac{125}{2}$ ft./sec.

Ex. 3. A point has equal velocities in two given directions. If one of these velocities be halved, the angle which the resultant makes with the other is halved also. Find the angle between the velocities.

Let $\overrightarrow{\mathrm{AB}}$ and $\overrightarrow{\mathrm{AD}}$ be two equal velocities inclined at an angle 2θ. The resultant velocity is given by $\overrightarrow{\mathrm{AC}}$.

The resultant of $\overrightarrow{\mathrm{AE}}\left(=\dfrac{\overrightarrow{\mathrm{AB}}}{2}\right)$, and $\overrightarrow{\mathrm{AD}}$ is $\overrightarrow{\mathrm{AF}}$. AF bisects the angle DAC. By a simple theorem of Geometry,

$\dfrac{\mathrm{AC}}{\mathrm{AD}}=\dfrac{\mathrm{CF}}{\mathrm{DF}}=1$; because F is the middle point of CD,

Therefore AC=AD.

[But $AC^2 = AB^2 + AD^2 + 2AB.AD.\cos 2\theta.$

$= 4AC^2 \cos^2\theta.$

or $\cos\theta = \frac{1}{2}$

$\therefore$ $\theta = 60°$ and

$\angle DAB = 120°$.]

Aliter. Since AB=AD and AC=AD,

$\therefore$ AB=BC=AC, and ABC is an equilalteral triangle.

$\therefore \angle CAB = 60°$ and

$\angle DAB = 120°$.

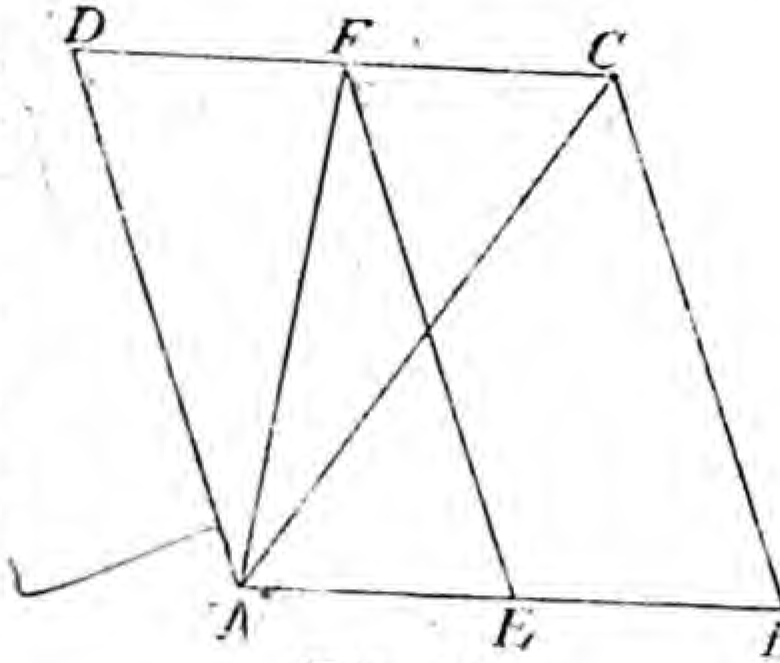


Fig. 15

Ex. 4. A stream flows with a velocity v ft./sec. Show that a swimmer, whose velocity in still water is c ft./sec., will take less time when he swims a distance d ft. and back across the current, than when he swims the same distance up and down the steam. $(c > v)$.

When flowing down the stream he will be helped by the stream and his velocity will be $c+v$. Therefore the time taken for swimming a distance d is $\dfrac{d}{c+v}$ secs.

Similarly, his velocity up the stream is $c-v$, and the time taken to swim the distance d is $\dfrac{d}{c-v}$ secs. Therefore the time for the up and down journey is

$$T = \frac{d}{c+v} + \frac{d}{c-v} = \frac{2cd}{c^2-v^2} \text{ secs} \quad \dots \quad \dots \quad (1)$$

Now, in order to swim from A to B, the swimmer must alway point in the direction AD, because the current will drag him in the direction DB. Therefore, his velocity along AB is $\sqrt{c^2-v^2}$. Hence to swim from A to B and then from B to A, he will take T_1 seconds.

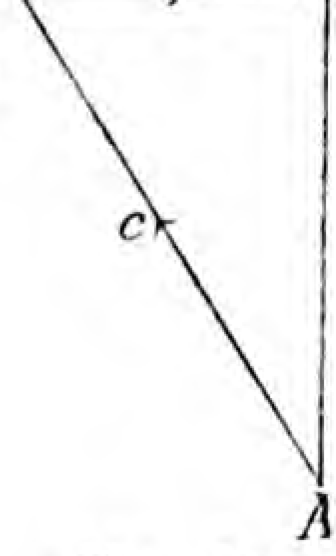


Fig. 16

$$T_1 = \frac{2d}{\sqrt{c^2-v^2}} = \frac{2d\sqrt{c^2-v^2}}{c^2-v^2}$$

$$= \frac{2cd\sqrt{1-\frac{v^2}{c^2}}}{c^2-v^2} \quad \dots \dots \quad (2)$$

From (1) and (2) it is clear that if $c > v$, then $T > T_1$.

Examples I

1. A particle O has three simultaneous velocities 1 ft./sec., 2 ft./sec. and 3 ft./sec. along the sides of an equilateral triangle, taken in order; find the magnitude and direction of the resultant velocity.

2. If a point has a velocity of 1 ft. per sec. to the east and also a velocity of $\sqrt{3}$ ft. per sec. to the north, determine the velocity which must be compounded with these to bring the point to rest.

3. A ship is sailing north at the rate of 4 ft. per sec.; the current is taking it east at the rate of 3 ft. per sec., and a sailor is climbing a vertical pole at the rate of 2 ft. per sec.; find the velocity and direction of the sailor in space.

4. If two velocities u and v are inclined at such an angle that their resultant is u, show that if u is doubled, the resultant of $2u$ and v is at right angles to v.

5. Find the components of a velocity of 100 ft. per second along two directions inclined at 30° and 45° respectively to its direction on opposite sides and in the same plane with it.

6. A train going at the rate of 45 miles an hour takes $\frac{1}{2}$ minute in passing another train 230 yds. long going in the same direction at the rate of 15 miles an hour. What is the length of the first train.

7. A man is in a boat at a point O on the bank of a river 90 feet broad; the stream flows with a velocity of 4 miles per hour and the man can row at 6 miles per hour; A is the point on the opposite bank perpendicularly opposite to O. In what direction must he row so as to take least time to cross the river, and find this time.

CHAPTER III

ACCELERATION

28. Change of Velocity. Instances of motion involving change of velocity are found everywhere in nature. A railway train does not attain full velocity immediately after the motion has started. The velocity slowly increases until after some time it becomes fairly constant. Similarly, a fast moving train does not immediately come to a standstill. The velocity slowly diminishes when the brakes are applied to the wheels.

The velocity of a body may be either uniform or variable. In the latter case the velocity may change in magnitude or direction, or in both.

If there is a change in the magnitude only and no change in direction, and if the initial velocity is u and the final velocity after a time t is v, then,

The change in velocity in time $t = v - u$.

When there is a change in direction also, suppose, that the initial velocity is represented by $\overrightarrow{AB}$ and the final velocity by $\overrightarrow{AC}$. Clearly, by the Law of Triangle of velocities $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

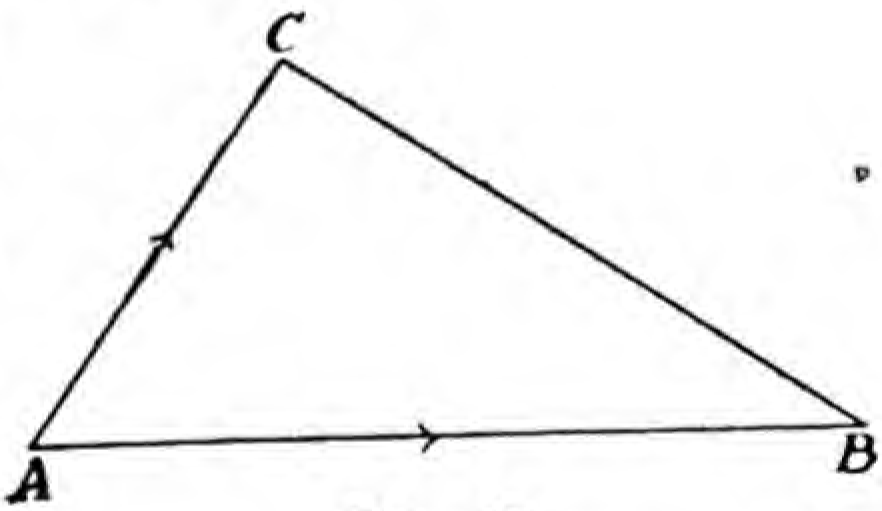


Fig. 17

Hence $\overrightarrow{BC}$ must be compounded with $\overrightarrow{AB}$ to give $\overrightarrow{AC}$. Therefore $\overrightarrow{BC}$ represents the change in velocity.

Hence, the change in velocity during a given time is that velocity, which, compounded with the initial velocity, is equivalent to the final velocity.

29. Acceleration. The acceleration of a moving point is the rate of change of its velocity.

It has magnitude, direction and sense, and is therefore a vector quantity. It obeys the Law of parallelogram of the combinations of vectors and also other theorems of vector algebra.

When the velocity of the particle moving in a straight line increases, the acceleration is **positive** and when it decreases, the acceleration is **negative**. The negative acceleration is also known as **retardation** or **deceleration**.

30. Uniform and Variable Acceleration. A particle is said to be moving with a **uniform acceleration**, if the change of its velocity is always in the same direction and is proportional to time ; *i.e.* if the change in velocity is equal in magnitude and direction in equal intervals of duration, however small or big the intervals may be.

The acceleration of a particle is **variable** if its magnitude or direction or both change with respect to time.

The motion of a particle in a straight line whose acceleration changes with respect to time furnishes the case, where, only the magnitude of acceleration is changing.

The motion of a particle in a circle with a uniform speed is an example in which the acceleration is ever changing its direction.

The motion of a particle with changes in both magnitude and velocity is furnished by the general motion of a particle along any curve.

31. Measurement and Unit of Acceleration. The uniform acceleration is measured by the change of velocity in unit time.

For variable acceleration it is necessary to mention the instant at which it has been measured. If u is the velocity at the beginning and v is the velocity at the end of time t, then, the ultimate value of $\left(\frac{v-u}{t}\right)$ as t is taken smaller and smaller, measures the acceleration *in the direction of motion*, at the instant under consideration.

In F. P. S. system the unit of acceleration is the acceleration of a particle whose velocity changes by 1 foot-sec., during every second, and is called one foot per second per second or 1 ft./sec.^{2}.

In C. G. S. system the unit is one cm. per second per second or 1 cm./sec.^{2}.

32. Theorem of Parallelogram of Accelerations. *If a moving point have simultaneously two accelerations represented in magnitude and direction by two sides of a parallelogram drawn from a point, they are equivalent to an acceleration represented by the diagonal of the parallelogram passing through that point.*

Let the accelerations be represented by the sides AB and AD of the parallelogram ABCD ; *i.e.* let $\overrightarrow{AB}$ and $\overrightarrow{AD}$ represent the velocities added to the velocity of the point in unit time. On the same scale let $\overrightarrow{EF}$ represent the velocity of the particle at some instant.

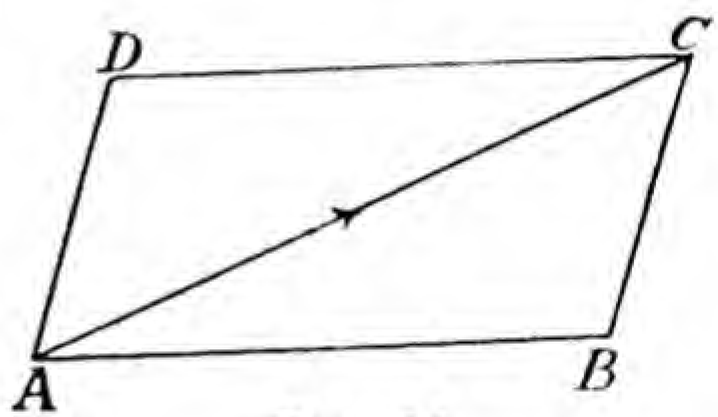


Fig. 18

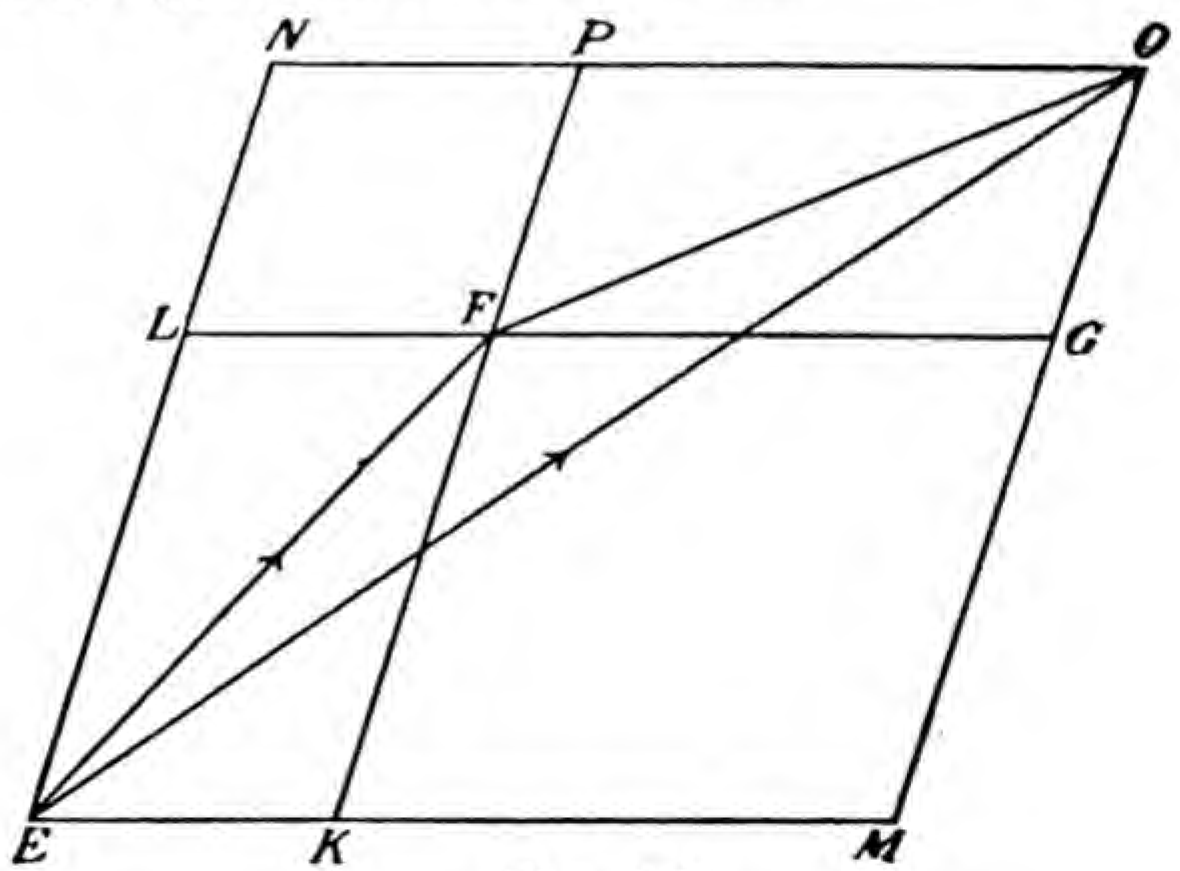


Fig. 19

Draw the parallelogram EKFL having its sides parallel to AB and AD, and complete the parallelogram EMON, with

$$AB = FG, \qquad AD = FP.$$

Initial velocity $\vec{EF} = \vec{EK} + \vec{EL}$.

After one second, the velocity components are $\vec{EM}$ and $\vec{EN}$; $\vec{KM} = \vec{AB}$ and $\vec{LN} = \vec{AD}$, being the accelerations or the changes of velocities in the directions EK and EL.

Therefore, the final velocity is

$$\vec{EM} + \vec{EN} = \vec{EO}.$$

Hence, by the triangle of velocities,

$$\vec{EF} + \vec{FO} = \vec{EO}.$$

that is, $\vec{FO}$ is the change of velocity.

But $\vec{FO} = \vec{AC}$, the diagonal of the parallelogram ABCD.

Hence the resultant of the accelerations $\vec{AB}$ and $\vec{AD}$ is the acceleration represented by $\vec{AC}$ both in magnitude and direction.

33. *A point moves in a straight line, starting with velocity u, and moving with constant acceleration f in its direction of motion; if v be its velocity at the end of time t, and s be its distance at that instant from its starting point, then*

(1) $v = u + ft$.

(2) $s = ut + \frac{1}{2} f t^2$.

(3) $v^2 = u^2 + 2 f s$.

1. u = initial velocity.

v = velocity at the end of time t.

f = acceleration, which is uniform.

Acceleration denotes the change in velocity in unit time. Hence in time t, the change in velocity is ft

$\therefore \quad v-u=$change in velocity in time $t=ft$

$$\therefore \quad v=u+f\,t. \qquad \text{.........(1)}$$

2. If s is the distance traversed in time t, and V the average velocity during the interval under consideration, then, we have

$$V=\frac{u+v}{2}=u+\frac{ft}{2} \qquad [\text{ from (1) }].$$

$$\therefore \quad s=V\,t=u\,t+\tfrac{1}{2}ft^2. \qquad \text{......(2)}$$

3. From (1) $\qquad t=\dfrac{v-u}{f}.$

Putting the value of t in (2), we get

$$s=\left(u+\frac{ft}{2}\right)t$$

$$=\frac{v+u}{2}\cdot\frac{v-u}{f}$$

or $v^2-u^2=2\,f\,s$

$$v^2=u^2+2\,f\,s.$$

Another proof of (3)

If V is the average velocity

$$V=\frac{v+u}{2}.$$

and the change in velocity

$$ft=v-u.$$

$$\therefore \quad t=\frac{v-u}{f}.$$

Now the distance travelled, *i.e.* $s=V\,t$

$$=\frac{v+u}{2}\cdot\frac{v-u}{f}$$

$$\therefore \quad v^2=u^2+2\,f\,s.$$

Cor. If the particle starts from rest, $u=0$ and therefore the above formulae become,

$$v=ft$$
$$s=\tfrac{1}{2}ft^2$$
and $$v^2=2fs.$$

34. *Space described in any particular second.*

If a particle starting from A reaches B in $(t-1)$ seconds and C in t seconds, then BC is the space described in the t^{th} second.

$$\begin{aligned} BC &= AC-AB \\ &= [ut+\tfrac{1}{2}ft^2]-[u(t-1)+\tfrac{1}{2}f(t-1)^2] \\ &= u[t-t+1]+\tfrac{1}{2}f[t^2-(t-1)^2] \\ &= u+\tfrac{1}{2}f(2t-1) \end{aligned}$$

Another method of calculating BC is as follows :—

The velocity at B is v_1,

$$v_1=u+f(t-1)$$

$\therefore$ $BC=v_1+\tfrac{1}{2}f$ [$t=1$, since BC is described in one second].

$$\begin{aligned} &= u+f(t-1)+\tfrac{1}{2}f \\ &= u+\tfrac{1}{2}f(2t-1). \end{aligned}$$

In the above formula putting $t=1, 2, 3\ldots.$ n we get the spaces described in the first, second, third,... and n^{th} seconds of the motion. The spaces are

$$u+\tfrac{1}{2}f,\ u+\tfrac{3}{2}f,\ldots\ldots\ldots \qquad u+\frac{2n-1}{2}f.$$

These distances form an arithmetical progression whose common difference is f.

Ex. Prove that if a particle move with uniform acceleration, the spaces described in consecutive equal intervals of time are in arithmetic progression.

Ex. 1. A point starts with a velocity of 100 cms. per second and moves with -2 cms./sec^2. acceleration. When will its velocity be zero, and how far will it have gone ?

$$v=u+ft$$
$\therefore$ $$0=100-2t$$
or $$t=50 \text{ seconds}.$$

Again

$$v^2 = u^2 + 2fs.$$
$$0 = 100^2 - 4s$$

or
$$s = 2500 \text{ cms.}$$

Ex. 2. A body starting from rest and moving with uniform occeleration, describes 171 feet in the 10^{th} second; find its acceleration.

$$s = ut + \tfrac{1}{2} ft^2.$$

Here u is zero. The space described in 9 seconds,
$$s_1 = \tfrac{1}{2} f . 81.$$

The space described in 10 seconds,
$$s_2 = \tfrac{1}{2} f . 100.$$

Therefore the space described in the tenth second,
$$s_2 - s_1 = 171$$

or
$$171 = \tfrac{1}{2} f(100 - 81),$$

or
$$f = \frac{2 \times 171}{19} = 18 \text{ ft./sec}^2.$$

Ex. 3. A train, which is uniformly accelerated, starts from rest and at the end of 3 secs. has a velocity with which it would traverse through 1 mile in the next minute; find the acceleration.

Let the uniform acceleration be f ft./sec^2. Then, the velocity at the end of 3 seconds is,

$$v = u + ft$$

or
$$v = 3f$$

Now the body travels 1760×3 feet in the next 60 seconds,
$$\therefore \quad s = ut + \tfrac{1}{2} ft^2,$$

$$\text{or } 1760 \times 3 = 3f \times 60 + \tfrac{1}{2} f . 3600$$

or
$$f = \tfrac{8}{3} = 2\tfrac{2}{3} \text{ ft./sec}^2.$$

Ex. 4. When a car is passing successive milestones its velocity is noted as being 10 mls./hr. and 15 mls./hr. If it is uniformly accelerated, find its velocity when passing the next milestone and the time taken.

In passing from one milestone to another, the velocity of the car changes from 10 mls./hr. to 15 mls./hr. Therefore the

acceleration is given by

$$v^2=u^2+2fs,$$
$$15^2=10^2+2f,$$

or
$$f=\frac{125}{2}\text{ mls./hr}^2.$$

Hence its velocity when passing the next milestone is

$$v^2=u^2+2fs$$
$$=15^2+2\times\frac{125}{2},$$

or
$$v=\sqrt{350}\text{ mls./hr.}=18{\cdot}7\text{ mls./hr. nearly.}$$

The time t taken to travel from the second to the third mile-stone is given by $v=u+ft$,

$$\sqrt{350}=15+\frac{125}{2}\,t,$$

$$\text{or}\quad t=\frac{3{\cdot}7\times 2}{125}\text{ hrs.}$$

$$=\frac{7{\cdot}4\times 60}{125}\text{ min.}=3{\cdot}6\text{ minutes nearly.}$$

Ex. 5. A train takes time T to perform a journey; it travels for time $\frac{T}{n}$ with uniform acceleration, then for time $\frac{(n-2)T}{n}$ with uniform speed V, and finally for time $\frac{T}{n}$ with constant retardation. Prove that the average speed is $\frac{n-1}{n}$ V.

Let S be the total distance travelled. The train starts from rest from O and due to uniform acceleration its velocity at A becomes V. Hence the average velocity for motion from O to A $\frac{V}{2}$. The velocity is V at B and because the distance BC is described in the same time as the distance AB, it is evident that the average speed for this part of the journey is also $\frac{V}{2}$.

Fig. 20

$$\therefore \qquad S=\frac{V}{2}.\frac{T}{n}+V\frac{(n-2)}{n}T.+\frac{V}{2}.\frac{T}{n}$$

$$=T.\frac{n-1}{n}V.$$

Hence the average velocity for the whole journey is

$$\frac{S}{T}=\frac{n-1}{n}V.$$

Ex. 6. Two trains, each 600 feet long, are running on straight parallel tracks in the same direction. At a certain instant one is travelling at 60 miles per hour with a deceleration of 0·25 ft./sec². and the other at 15 miles per hour with an acceleration of 0·5 ft./sec.² and the head of the faster train is just level with the tail of the slower one.

How long will it take the faster train just to clear the slower one ?

Suppose AB and CD represent the positions of the fast and slow trains respectively. After time t the positions are denoted by A_1B_1 and C_1D_1, when the fast train has just

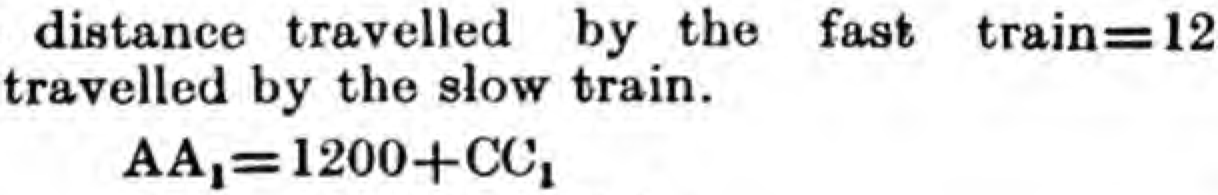


Fig 21

cleared off the slow train.

The distance travelled by the fast train=1200+The distance travelled by the slow train.

or $\qquad AA_1=1200+CC_1 \qquad$ ......(1)

But $\qquad AA_1=88t-\frac{1}{2}(0·25)t^2$

and $\qquad CC_1=22t+\frac{1}{2}(0·5)t^2$

Substituting in (1) we get

$$88t-\tfrac{1}{8}t^2=1200+22t+\tfrac{1}{4}t^2$$

or $t^2-176t+3200=0$ ......(2)

Therefore $t=20·65$ seconds or 155·35 seconds.

After 20·65 seconds approximately the train AB will just clear the train CD. *i.e.,* the tail of AB, the point B_1, will just

coincide with the head of CD, the point C_1. As time passes after this occurrence, the train AB for some time will remain ahead of the train CD. But due to deceleration the velocity of AB is constantly diminishing, while that of CD is constantly increasing. So after 155·35 seconds, once again the tail of AB will just coincide with the head of CD.

Examples II

1. A train is moving with a uniform speed of 45 miles per hour and the brakes produce a retardation of 4 ft./sec². At what distance from the station should the brakes be applied so that the train may stop at the station ?

2. A train starting from rest moves with uniform acceleration and describes the first mile in two minutes. If it now moves on with a uniform velocity, how long will it take to describe another mile ?

3. A body moves over 30 feet during the 5th second and 42 feet during the 7th second of its motion. Find the space passed over in 10 seconds.

4. A bullet passes through a wall 9 inches thick. If its velocity changes from 1,000 to 600 ft./sec., find the retardation produced by the resistance of the wall and the time occupied in passing through it.

5. A bullet strikes a sand-bag with a velocity of 1200 ft./sec. and penetrates a distance of 4 feet. Find its velocity after penetrating 2 feet, assuming the retardation to be uniform.

6. A bullet fired into a target loses half its velocity after penetrating 3 inches. How much farther will it penetrate ?

7. A ball rolling down a slope with uniform acceleration passes a series of posts driven in the ground at intervals of 10 feet. The velocities when passing three successive posts are v_1, v_2, v_3. Prove that $v_1^2+v_3^2=2v_2^2$.

8. A particle starts with a velocity of 200 cms. per second and moves in a straight line with a retardation of 10 cms./sec²; find how long elapses before it has described 1500 cms. and explain the double answer.

9. Two points move in the same straight line starting at the same moment from the same point in it ; the first moves with constant velocity u and the second with constant acceleration f; during the time that elapses before the second catches the first show that the greatest distance between the particles is $\frac{u^2}{2f}$ at the end of time $\frac{u}{f}$ from the start.

10. A car going along a straight road and moving with uniform deceleration traverses three successive stretches A, B and C each of length 210 feet. Three seconds are taken over the stretch B and four seconds over C. Find the time taken over the stretch A.

11. For $\frac{1}{m}$ of the distance between two stations, a train is uniformly accellerated, and for $\frac{1}{n}$ of the distance it is uniformly decelerated ; it starts from rest at one station and comes to rest at the other. Prove that the ratio of the greatest velocity to its average velocity is $1+\frac{1}{m}+\frac{1}{n}:1$.

12. If the distances gone over by a point moving with uniform acceleration in the pth, qth, rth seconds of its motion are, respectively, x, y, and z, show that

$$(q-r)x+(r-p)y+(p-q)z=0.$$

13. The speed of a train increases at a constant rate α from 0 to v, then remains constant for an interval, and finally decreases to 0 at a constant rate β. If l be the total distance described, prove that the total time occupied is

$$\frac{l}{v}+\tfrac{1}{2}v\left(\frac{1}{\alpha}+\frac{1}{\beta}\right).$$

35. Relative Velocity. We have already introduced the concept of relative velocity (Art. 19), by considering the example of two trains moving with different velocities along parallel tracks. Here, we present the same concept in a slightly generalized form.

Suppose, a body A is moving along OC with a velocity u, whilst another body B is moving along O_1D inclined at an angle θ to OC.

Then, $v\cos\theta$ and $v\sin\theta$ are the resolved parts of the velocity of B along OC and perpendicular to OC.

The velocity of B relative to A along OC is $v\cos\theta-u$; and since A has no velocity perpendicular to OC, the velocity of B relative to A in that direction is $v\sin\theta$.

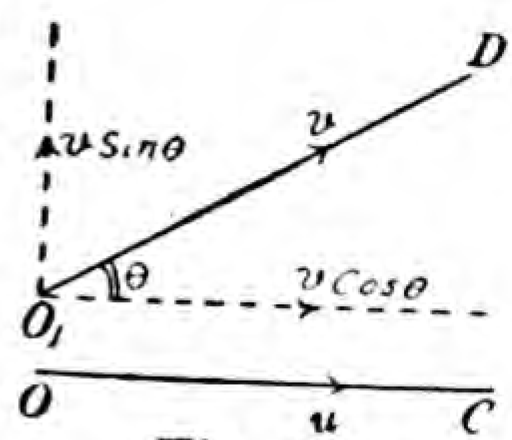


Fig. 22

Hence the velocity of B relative to A consists of two components ; $v\cos\theta-u$ along OC and $v\sin\theta$, in the direction perpendicular to OC.

Relative velocity of $B=\sqrt{(v\cos\theta-u)^2+v^2\sin^2\theta}$

$$=\sqrt{v^2+u^2-2uv\cos\theta}$$

Working Rule. *In order to find the velocity of B relative to A, combine with the velocity of B a velocity that is equal but opposite to that of A. The resultant of these two latter velocities gives the required relative velocity of B.*

36. Geometrical approach. Let AP and BQ represent the actual velocities of A and B. Complete the parallelogram A B R P. Join RQ. By the theorem of the triangle of velocities, the velocity BQ is equivalent to two velocities represented by BR and RQ. *i.e.*,

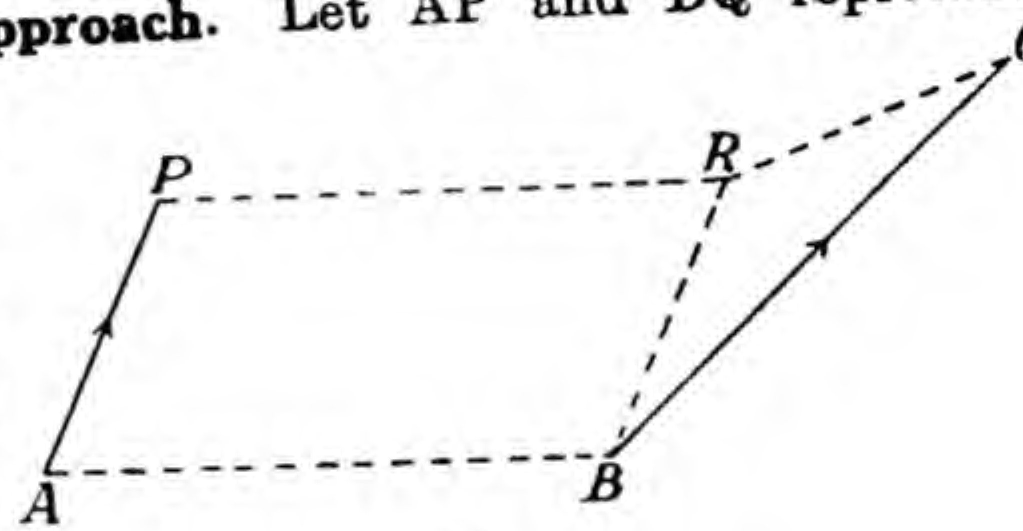


Fig. 23

$$\vec{BQ}=\vec{BR}+\vec{RQ}$$

Relative velocity of $B=\vec{BQ}-\vec{AP}$

$$=\vec{BR}+\vec{RQ}-\vec{AP}$$

$$=\vec{RQ}\ ;\quad \text{Since } \vec{BR}-\vec{AP}=0.$$

Ex. 1. A steamer is going due west at 14 miles per hour, and the wind appears from the drift of the clouds to be blowing at 7 miles per hour from the north-west. Find the actual velocity of the wind and make a geometrical construction for its direction.

The relative velocity of the wind is the resultant of the actual velocity of the wind and a velocity equal and opposite to that of the steamer.

Take 14 m. p. h. along OX. Produce CO to D, making OD equal to 7 m. p. h. Complete the parallelogram OADE. Then O E gives the actual velocity of the wind.

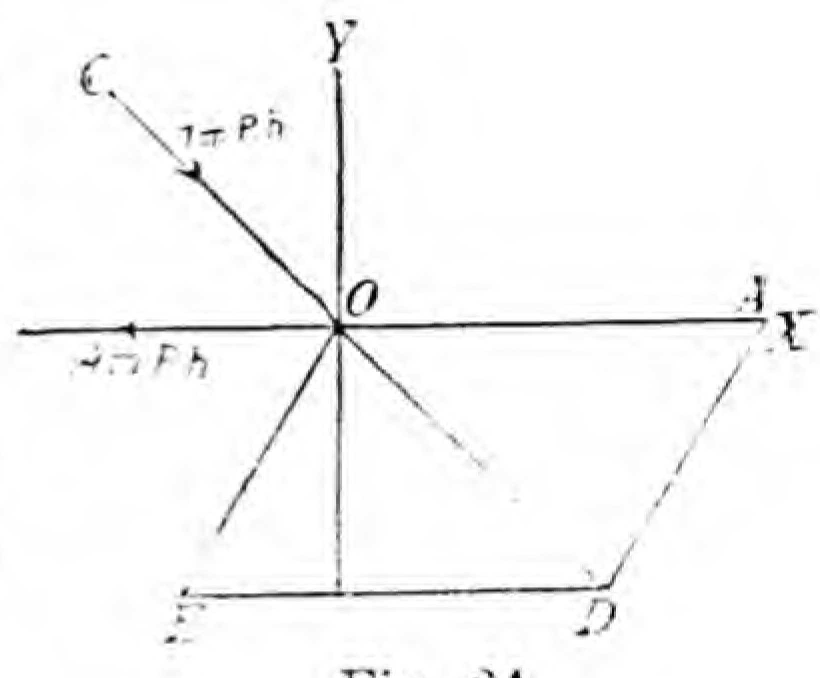


Fig. 24

In the $\triangle OAD$

$$AD = \sqrt{OA^2 + OD^2 - 2OA.\,OD\ \cos DOA}$$

$$= \sqrt{14^2 + 7^2 - 2.\ 14.\ 7.\ \frac{1}{\sqrt{2}}}$$

$\therefore$ Actual velocity of the wind $= 7\sqrt{5 - 2\sqrt{2}}$.

Ex. 2. An aeroplane's speed (*i.e.*, velocity relative to the air) is represented in the figure by the length AB. The pilot finds that when the plane is pointing northwords, he is actually travelling in the direction AC, and that when it is pointing southwards, he is travlling in the direction BC. Show that the velocity of the wind is represented in magnitude and direction by 2MC, where M is the middle point of AB.

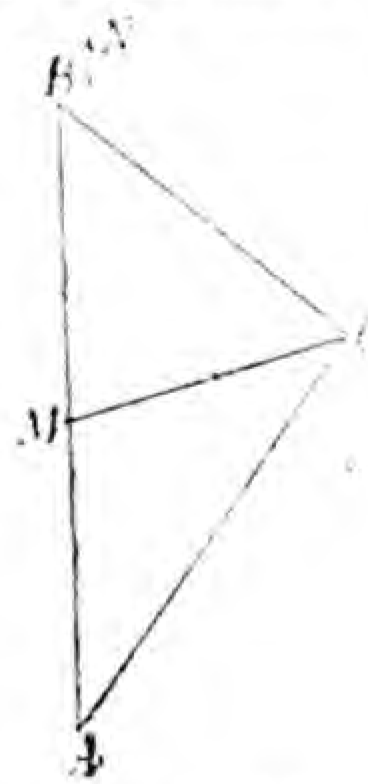


Fig. 25

Let BK be the actual velocity of the wind. Then

$$\overrightarrow{AB} + \overrightarrow{BK} = \overrightarrow{AK},$$

where AK is obtained by producing AC to K.

Again

$$\overrightarrow{BA}+\overrightarrow{BK}=\overrightarrow{BE},$$

where BE is the diagonal of the parallelgram A B K E. Hence BC must coincide with the line BE, or C is the middle point of AK.

In the $\triangle ABK$, join C to M, the middle point of AB ; Then MC is parallel and half of BK.

Therefore $\overrightarrow{BK}=2\overrightarrow{MC}$.

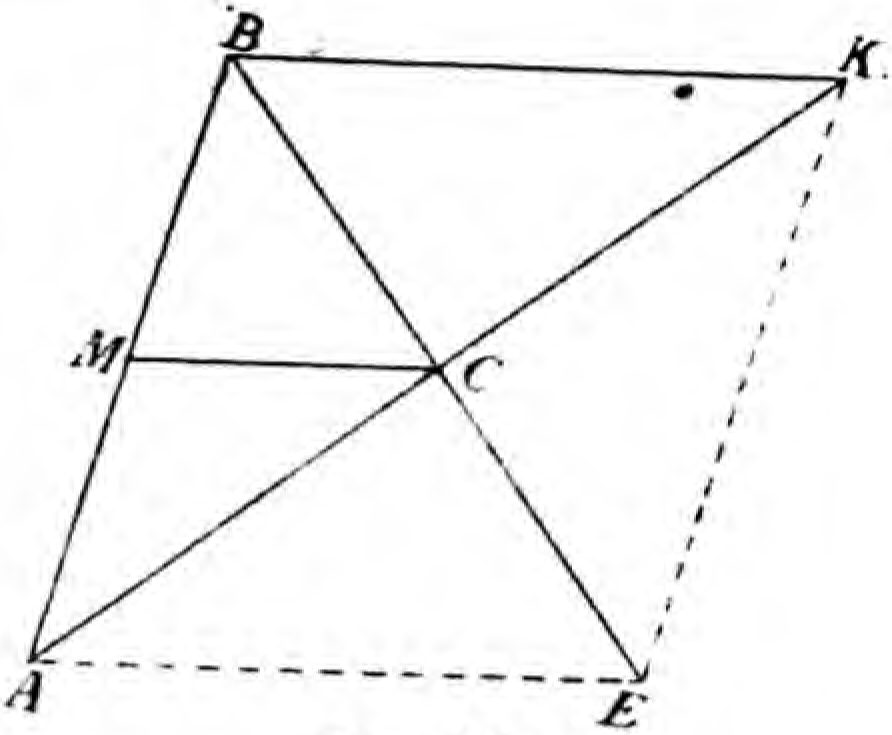


Fig. 26

Ex. 3. A boat takes 10 minutes to cross a straight river in a straight line from a point A on one bank to a point B on the other bank, and 20 minutes to do the return journey. The current flows at 3 miles per hour and the speed of boat relative to water is 6 miles per hour.

Find the width of the river and the downstream distance from A to B.

Let $AC=a$ be the width of the river and $AB=b$, the distance downstream from A to B.

Let θ and ϕ be the angles which the resultant direction makes with the stream in the first case and the second case.

The components of the velocity of the boat along and perpendicular to the stream are

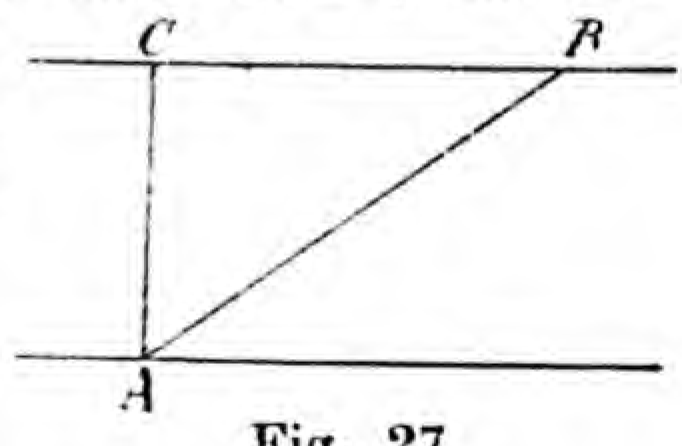


Fig. 27

$$6\cos\theta+3 \text{ and } 6\sin\theta \qquad \text{m.p.h.}$$

while, the components of velocity in the second case, when the boat is making its return journey, up the stream and perpendicular to it, are

$$6\cos\phi-3 \text{ and } 6\sin\phi \qquad \text{m.p.h.}$$

In the first case the boat takes $\frac{1}{6}$ hours and in the second case it takes $\frac{1}{3}$ hours to complete its journey. Hence

$$a=\tfrac{1}{6}\times 6\sin\theta=\tfrac{1}{3}\times 6\sin\phi \qquad ...(1)$$

or $\qquad a=\sin\theta=2\sin\phi \qquad \ldots(2)$

Also $\qquad b=\frac{1}{6}(6\cos\theta+3)=\frac{1}{3}(6\cos\phi-3)$

or $\qquad b-\frac{1}{2}=\cos\theta \qquad \ldots(3)$

$\qquad b+1=2\cos\phi \qquad \ldots(4)$

To the square of (2) adding the square of (3)

$$a^2+(b-\tfrac{1}{2})^2=\sin^2\theta+\cos^2\theta=1 \qquad \ldots(5)$$

Again from (2) and (4)

$$a^2+(b+1)^2=4(\sin^2\phi+\cos^2\phi)=4 \qquad \ldots(6)$$

Subtracting (5) from (6)

$$3b+\tfrac{3}{4}=3 \qquad \text{or} \quad b=\tfrac{3}{4}\text{ miles.}$$

$$\therefore \qquad a=\sqrt{1-\tfrac{1}{16}}=\frac{\sqrt{15}}{4}\text{ miles.}$$

Examples III

1. Suppose a flying machine travels in still air at 100 miles an hour. How long will it take the aviator to fly round a square course whose side is 6 miles long if there is a wind blowing parallel to a pair of sides at the rate of 28 miles an hour.

2. How long would the aviator mentioned in the above question take to fl round the same square course if the wind were blowing parallel to a diagonal of the square ?

3. A man tricycling at the rate of 8 miles an hour due west, feels the wind to be blowing from the south ; on increasing his speed to 16 miles an hour the wind appears to be blowing from the south-west ; find the velocity of the wind.

4. Two trains, each 200 feet long, are moving towards each other on parallel lines with velocities of 20 and 30 miles per hour respectively. Find the time that elapses from the instant when they first meet until they have cleared each other.

5. Two trains, start at the same instant from a station and run along straight lines which make an angle of 60° with each other, at the rate of 30 and 40 miles an hour respectively. Find the relative velocity in direction and magnitude.

6. Two persons are walking along two straight roads perpendicular to each other, with velocities 8 and 6 miles an hour respectively. If at a given instant, both of them be at a distance of 2 miles from the crossing of the roads and walk towards it, how long after this will the distance between them be least and what is the least distance ?

7. A railway train is moving at the rate of 28 miles per hour, when a pistol shot strikes it in a direction making an angle $\sin^{-1}\frac{3}{5}$ with the

train. The shot enters one compartment at the corner furthest from the engine and passes out diagonally opposite corner; the compartment being 8 feet long and 6 feet wide, show that the shot is moving at the rate of 80 miles an hour, and traverses the carriage in $\frac{5}{44}$ths of a second.

8. If an aviator flies round a triangular course, each side of which is c miles long, while the wind blows parallel to one side at u miles per hour, show that he takes

$$c\left\{\frac{v+\sqrt{4v^2-3u^2}}{v^2-u^2}\right\} \text{ hours to complete the circuit}$$

in either direction, v being his velocity relative to the air.

9. Two particles start from rest and move along two lines inclined at an angle α. One moves with a uniform velocity v and the other with a uniform acceleration f. Show that their relative velocity is least after a time $\frac{v\cos\alpha}{f}$, and find its value.

CHAPTER IV

VERTICAL MOTION UNDER GRAVITY

37. Historical. Classification of knowledge is one of the greatest achievements of human mind. The followers of Aristotle (460 – 370 B.C.), a Greek philosopher and experimenter, explained the descent of heavy bodies and the rising of light bodies by assuming that every thing sought its place : the place of heavy bodies was below, the place of light bodies was above. Motions were divided into natural motions, as that of descent, and violent motions, as, for example that of a projectile. From some experiments and observations, philosophers had concluded that heavy bodies fall more quickly and lighter bodies more slowly. Such ideas prevailed at the time when Galileo took up the study of the falling bodies. He asked the question, "*How* do heavy bodies fall ?" As a trial hypothesis he assumed that the velocities acquired in the descent increase proportionally to the distance descended through. He argued, that if a body had acquired a certain velocity in the first distance descended through, double the velocity in double such distance descended through, and so on ; that is to say, if the velocity in the second instance is double that in the first, then the double distance will be travelled in the same time as the original simple distance. If accordingly, in the case of the double distance we conceive the first half traversed, no time will, it would seem, fall to the account of the second half. The motion of a falling body appears, therefore, to take place instantaneously, which contradicts the hypothesis proposed.

As a second trial hypothesis Galileo assumed that the velocity acquired is proportional to the time of descent. If a body fall once, and then fall again during twice as long an interval of time as it fell in the first instance, it will attain in the second instance double the velocity it acquired in the first. This hypothesis also could not stand the test of experiment.

Galileo as a result of his experiments on the moving bodies found, that the time of descent, the final velocity and the distance traversed are related in a definite manner. If 'g' denotes the acceleration due to gravity, the following table serves to bring out the relationship. If the particle start from rest and fall freely under gravity, we have

t=time of descent; v=velocity acquired; s=distance traversed.

t.	v.	s.
1	$1.g$	$1\times 1.\frac{g}{2}$
2	$2.g$	$2\times 2.\frac{g}{2}$
3	$3.g$	$3\times 3.\frac{g}{2}$
4	$4.g$	$4\times 4.\frac{g}{2}$
...	...	...
...	...	...
t	$t.g$	$t\times t.\frac{g}{2}$.

From this table it is obvious, that,

$$v=gt.$$
$$s=\tfrac{1}{2}\ gt^2.$$

These results have already been arrived at in the previous chapter. They are particular cases of equations (1), (2) and (3), Art. 33.

38. Experimental evidence. At the time of Galileo there did not exist any suitable instrument for measuring time. People usually measured time with the help of sun dials. Galileo used a water clock. It consisted of a vessel of water of very large transverse dimensions, having in the bottom a minute orifice which was closed with the finger. At the beginning of an interval, the orifice was opened and the water allowed to run in a vessel. At the end of the interval, the orifice of the clock was closed and the water which had

collected during that interval of time was weighed on a balance. Since the height of the liquid in the vessel did not change appreciably, the weights of the water discharged from the orifice were proportional to times. Galileo is also said to have employed the beat of the pulse for comparing intervals of time.

About the year 1590, Galileo showed that the acceleration of a falling body is constant.

Experiment. Take a big jar and remove all air from its interior by means of an exhaust pump. Then release a heavy piece of metal and a light cork simultaneously from the top of the jar. It is found that both of them strike the floor at the same time. This can only heppen if the accealeration acting on the falling bodies is constant.

Morin's Experiment. A cylinder is covered with paper and made to rotate uniformly about its axis which is vertical. In front of the cylinder is an iron weight, carrying a pencial P, which is compelled to fall freely in a vertical line. The falling pencil leaves a mark on the revolving cylinder. When the pencil has descended through a certain distance, the paper is taken out of the cylinderical roller, and stretched out on a flat surface. The curve marked out by the pencil is such that the vertical distances described by the pencil from the beginning of the motion are always proportional to the squares of the horizontal distances described by it, so that, if Q, R be any two points on the curve, then

$$\frac{AM}{AN}=\frac{QM^2}{RN^2}.$$

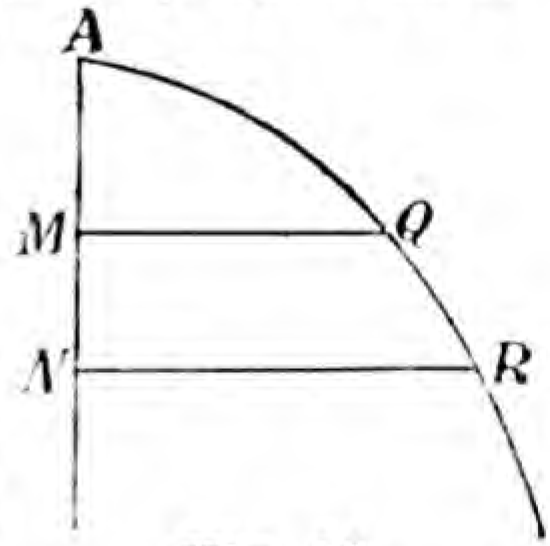


Fig. 28

Now since the cylinder revolved with a uniform velocity, these horizontal distances are proportional to the times that have elapsed from the commencement of the motion. Hence the vertical distance described is proportional to the square of the time from the commencement of the motion.

39. The acceleration due to gravity 'g' is different at different places on the surface of the earth. It depends on

the distance of the place from the centre of the earth. When foot-second units are employed, the value of g varies from about 32·091 at the equator to about 32·252 at the poles. In the C.G.S. units the extreme limits are about 978 and ·983 respectively.

40. Vertical Motion under Gravity. A body is projected upwards from the surface of the earth along a vertical line with a velocity u. In its motion of ascent the acceleration due to gravity 'g', is acting downwards, opposite to the direction of its motion. Hence the acceleration in the direction of motion is '$-g$'. This negative acceleration or retardation diminishes the upward velocity of the particle, so that after some time, the particle comes to rest for an instant and then immediately begins falling downwards, and retraces its path to the point of projection.

(*i*) *Time to attain a given height.* A particle is projected upwards along a vertical direction with the velocity u. An acceleration $-g$ is constantly acting on it. The time for attaining a height h is

$$h = ut - \tfrac{1}{2} g t^2.$$

This is a quadratic equation in 't' with both roots positive; the lesser root gives the time at which the particle is at the given height on its way up and the greater root gives the time at which it is at the same height on its way down.

(*ii*) *Greatest height attained.* The velocity at a given height h is

$$v^2 = u^2 - 2\,g\,h.$$

This velocity is independent of the time from start, and is therefore the same at the same point whether the body is going upwards or downwards.

At the highest point the velocity v is just zero. Therefore

$$0 = u^2 - 2\,g\,x.$$

Hence the greatest height $= \dfrac{u^2}{2g}$.

Also the time for the greatest height, T, is given by

$$0 = u - g\,\mathrm{T}.$$

or $$T=\frac{u}{g}.$$

(*iii*) *Velocity due to a free fall.* When a particle falls freely from rest under gravity, its initial velocity is zero and a constant acceleration g acts on it in its direction of motion. The velocity after falling through a distance, h, is

$$v^2=2\,g\,h,$$

or $$v=\sqrt{2\,g\,h}.$$

40. (*a*) **Relative Motion of two falling bodies**. Since the acceleration of gravity is the same for all bodies, *the relative acceleration of two bodies under gravity (being the difference of their actual accelerations) is zero.*

Hence their relative velocity is constant. This principle is of great importance in finding when and where two bodies projected in the same vertical line would meet, or in finding their distances apart at any given instant of time.

Ex. If a stone is dropped from a tower 100 ft. high, and another projected at the same instant from below with initial velocity $u=80$ ft./sec., find when and where they meet ?

Initially the velocities are 0 and 80 ft./sec., hence the lower one approaches the upper with a relative velocity $=80$ ft./sec., and since both have the same acceleration—that due to gravity,—their relative velocities remain constant. But their original distance is 100 ft.,

$$\therefore \text{ Time taken to meet} = \frac{100}{80}=1\tfrac{1}{4}\text{ sec.}$$

In this time upper stone would have fallen :—

$$s=\frac{1}{2}.\,32.\,\left(\frac{5}{4}\right)^2=25\text{ ft.}$$

Examples

Ex. 1. A stone is thrown vertically into a mine-shaft with a velocity of 96 feet per second, and reaches the bottom in 3 seconds ; find the depth of the shaft.

Since the particle is moving towards the earth, the acceleration is 'g' in the direction of motion.

$$\begin{aligned} s &= u\,t + \tfrac{1}{2}\,g\,t^2 \\ &= 96\times 3 + \tfrac{1}{2}\times 32\times 9 \\ &= 432 \text{ feet.} \end{aligned}$$

Ex. 2. A body falls freely from the top of a tower, and during the last second of its motion falls $\frac{16}{25}$ ths of the whole distance. Find the height of the tower.

Let s be the height of the tower, and t be the time taken to fall through a distance s. The distance moved in $(t-1)$ seconds is $s - \frac{16}{25}s = \frac{9}{25}s$

$$\therefore \quad \tfrac{9}{25}\,s = \tfrac{1}{2}\,g\,(t-1)^2 \qquad \ldots\ldots(1)$$

Also the distance moved in t seconds is s

$$s = \tfrac{1}{2}\,g\,t^2 \qquad \ldots\ldots(2)$$

Therefore dividing (1) by (2), we get

$$\frac{9}{25} = \frac{(t-1)^2}{t^2}$$

$$9t^2 = 25(t-1)^2$$

$$3t = 5(t-1)$$

or $$t = \tfrac{5}{2} \text{ seconds.}$$

Substituting in (2) we get,

$$s = \tfrac{1}{2}\times 32\times \tfrac{25}{4} = 100 \text{ feet.}$$

Particles dropped from a body in motion. When a particle is dropped from a body which is in motion, the initial velocity of the particle is that of the moving body. After this instant the particle falls under a constant acceleration due to gravity.

Ex. 3. From a balloon, ascending with a velocity of 32 ft./sec. a stone is dropped and reaches the ground in 17 seconds; how high was the balloon when the stone was dropped ?

If the distance is measured positively in the downward direction the initial velocity in the direction, of the distance increasing, is $-32'$/sec. If h be the height of the balloon at the time when the stone was dropped.

B. $\uparrow$ *32 ft./sec.*

Earth

$$h = ut + \tfrac{1}{2}gt^2$$
$$= (-32) \times 17 + \tfrac{1}{2} \times 32 \times 17^2$$
$$= 4080 \text{ feet.}$$

Ex. 4. A stone is dropped from a balloon which is rising with acceleration α, and t_1 seconds after this a second stone is dropped. Prove that the distance between the stones at a time t after the second is dropped is $\frac{1}{2}(\alpha+g)\, t_1(t_1+2t)$.

Impose upon the whole system (balloon, stone etc.) an acceleration α in the downward direction. The stone will then have an acceleration $(\alpha+g)$ in the downward direction and the balloon will be at rest.

Distance for the first stone from the balloon is

$$s_1 = \tfrac{1}{2}(\alpha+g)(t+t_1)^2$$

and distance for the second stone is

$$s_2 = \tfrac{1}{2}(\alpha+g)t^2$$

∴ the distance between them

$$= s_1 - s_2 = \tfrac{1}{2}(\alpha+g)t_1(t_1+2t).$$

Examples IV

1. A body is thrown vertically upwards with a velocity of 160 feet per second. How high will it rise? How long will it be before it returns to the point of projection?

2. A stone thrown vertically upwards is observed to pass upwards through a point P, and after an interval of 2 seconds, to pass downwards through the same point. Find the velocity of the stone at P.

[**Hint.** Imagine the stone to be thrown up from P with a velocity u. Since it comes back to the same point after two seconds, the time to reach the highest point is one second.]

3. A particle falls from rest and in the last second of its motion passes through 224 feet. Find the height from which it fell, and the time of its fall.

4. A body falls freely from the top of a tower, and during the last second it falls through $\frac{9}{25}$ of the whole distance. Find the height of the tower.

5. A stone is thrown vertically upwards with a velocity of 96 feet per second ; find how high it will rise. After 4 seconds from the projection of A, another stone B is let fall from the same point. Show that A will overtake B after 4 seconds more.

6. A tower is 64 feet high. From its foot a particle is thrown vertically up with a velocity of 64 ft./sec. and at the same instant another particle is dropped from the top of the tower. What time after will they meet ?

If the second particle be dropped one second later, what time will elapse before they meet ?

7. A balloon ascends with a uniform acceleration of 4 feet per sec. per sec. ; at the end of half a minute a body is released from it ; find the time that elapses before the body reaches the ground.

8. After a ball has been falling under gravity for 5 seconds, it passes through a pane of glass and loses half its velocity ; if it now reaches the ground in one second, find the height of the glass above the ground.

9. A ball is thrown vertically upwards with velocity g, and one second later another ball is thrown up from the same point with velocity $2g$. When and where will it strike the first ball ?

10. A lift ascending from a point 600 ft. deep rises during the first part of its ascent with uniform acceleration. On nearing the top the upward force is cut off, and the velocity of the lift is just sufficient to carry it to the top. If the whole process occupies 30 seconds, find the acceleration during the first part of its ascent and the maximum velocity attained.

11. A balloon is rising with a uniform velocity of 50 ft./sec and a stone projected vertically upwards from it reaches the ground in 10 seconds ; find the height of the balloon (1) when the stone was projected ; (2) when the stone reached the ground ; find also the greatest height attained by the stone.

12. Three particles are thrown vertically downwards with initial velocities u_1, u_2, u_3 from heights s_1, s_2, s_3 respectively, and they reach the ground simultaneously ; prove that

$$\frac{s_1 - s_2}{u_1 - u_2} = \frac{s_2 - s_3}{u_2 - u_3} = \frac{s_3 - s_1}{u_3 - u_1}.$$

13. A particle falls through x feet at two different places on the earth's surface and it is observed that at one place the time of falling is n seconds less, and the velocity acquired is m ft./sec. greater than at the other ; show that if the acceleration due to gravity at two places be g_1 and g_2 respectively, then $\frac{m}{n} = \sqrt{g_1 g_2}$.

41. Motion down a smooth inclined plane. Let AB be the vertical section of a smooth inclined plane inclined at an angle α to the horizon, and let P be a body on the plane.

The vertical acceleration g may be resolved along and perpendicular to the inclined plane.

The resolved part along the plane in the downward direction is g sinα. The resolved part perpendicular to the plane is g cos α.

The plane prevents any motion perpendicular to itself and therefore the body slides down the plane with an acceleration g sin α.

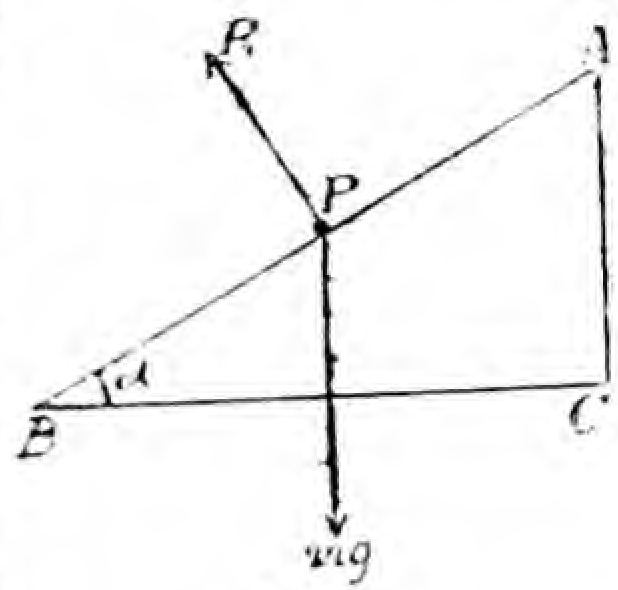


Fig.

If the particle P started from rest from A, the velocity acquired after sliding down a length l of the plane is

$$v^2 = 2.g \sin \alpha. \quad l = 2g.l \sin \alpha.$$

or $$v = \sqrt{2g.\, l \sin \alpha}$$

$$= \sqrt{2g.\, AC}.$$

The velocity acquired is independent of the angle which the lane makes with the horizon, and depends only on the vertical ight through which the particle has fallen.

42. If the body be projected up the plane with a velocity , its motion will be retarded. The acceleration in the direction of motion is $-g$ sin α. The distance travelled up the plane is given by

$$v^2 = u^2 - 2.\, g \sin \alpha.\, x$$

At the highest point $v = 0$

$$\therefore \quad x = \frac{u^2}{2g \sin \alpha}.$$

The time taken to traverse this distance is $0 = u - g \sin \alpha.\, t$

$$\therefore \quad t = \frac{u}{g \sin \alpha}.$$

43. Theorem. *The time that a body takes to slide down any smooth chord of a vertical circle, which is drawn from the highest point of the circle is constant.*

Let AB be the diameter of a vertical circle with A as the highest point and AD any chord.

Let $\angle DAB=\theta$; $AD=x$, $AB=a$ so that, $x=a\cos\theta$.

The component of acceleration down the chord AD is $g\cos\theta$. Let T be the time from A to D. Then AD is the distance described in time T by a particle starting from rest from A and moving with an acceleration $g\cos\theta$.

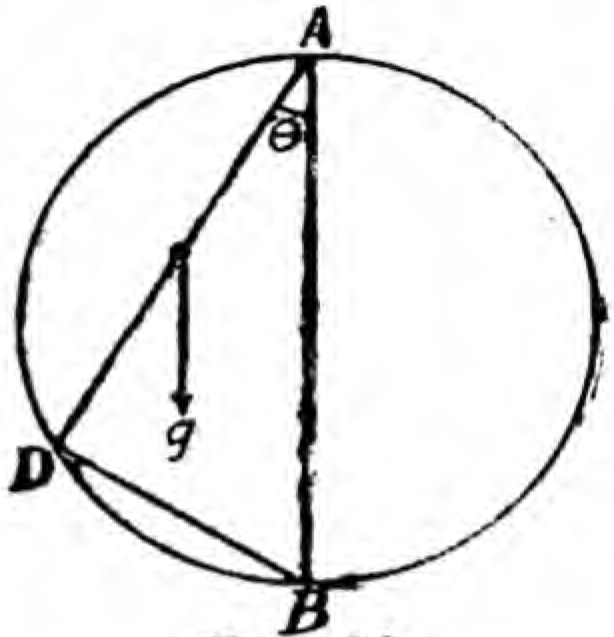


Fig. 30

$\therefore \quad x=\frac{1}{2}g\cos\theta.\ T^2$

$$\therefore \quad T=\sqrt{\frac{2x}{g\cos\theta}}=\sqrt{\frac{2a\cos\theta}{g\cos\theta}}=\sqrt{\frac{2a}{g}}.$$

The value of T is independent of θ and is the same as the time of falling vertically through the distance AB.

Hence the time of falling down all chords of the cirol beginning at A is the same.

44. Theorem. *The time that a body takes to slide dou any smooth chord of a vertical circle. which is drawn from any point of the circle ending in its lowest point, is constant.*

Let B be the lowest point of a verticle circle, with AB as its vertical diameter. BD is any chord always passing through B.

Let $\angle DBA=\theta$, $BD=x$. $AB=a$

$\therefore \quad BD=x=a\cos\theta.$

The component of acceleration along DB is $g\cos\theta$. The particle starts from rest from B and moves with acceleration $g\cos\theta$. If T is the time taken to describe the distance DB

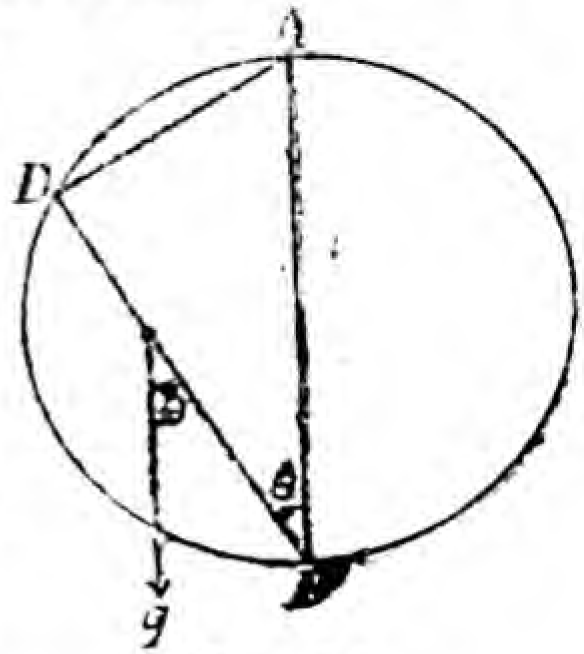


Fig. 31

$x=\frac{1}{2}g\cos\theta.\ T^2$

$$\therefore \quad T=\sqrt{\frac{2x}{g\cos\theta}}=\sqrt{\frac{2a\cos\theta}{g\cos\theta}}=\sqrt{\frac{2a}{g}}.$$

45. *To find the line of quickest descent from a given point to a given straight line.*

Let A be the given point, and BC the given straight line. Through A draw AD perpendicular to BC, and AE vertically down. Bisect $\angle$DAE by AF, cutting BC in F. Then AF shall be the line of quickest descent; that is, a particle would slide more quickly down AF than down any other line joining A to BC.

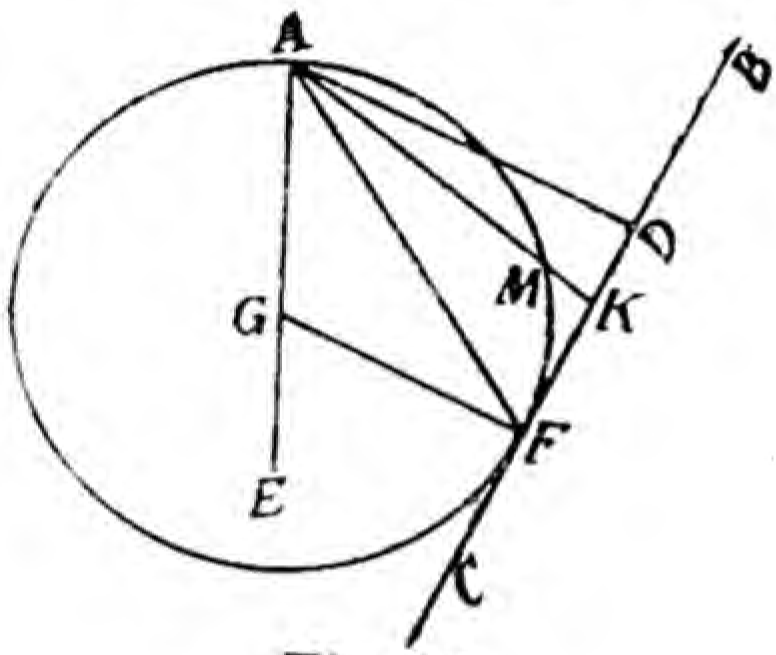


Fig. 32

Draw FG perpendicular to BC, cutting AE in G. Then, since FG and AD are parallel.

$\angle$AFG $=\angle$DAF.

But $\angle$DAF $=\angle$GAF, (construction)

$\therefore$ $\angle$GFA $=\angle$GAF

$\therefore$ GA $=$ GF.

Then, if with G as centre and GA as radius, we describe a circle, it will touch BC at F, and A will be its highest point.

Draw any other line AK cutting the circle in M. By Art. 41, the particle starting from rest from A would take the same time in sliding down AF as it would take for sliding down the chord AM, and hence this time is less than the time to slide down AK.

46. *To find the line of quickest descent from a point to a circle.* Let A be the given point, BCF the given circle and O its centre. Draw a radius OC, vertically down. Join AC, cutting the circumference in F; then AF shall be the line of quickest descent from A to the given circle.

Join OF, and produce it to meet the vertical through A in E.

Then $\angle$AFE $=\angle$OFC

and $\angle$OCF $=\angle$EAF (alternate angles, AE $\parallel$ OC)

Hence AE=EF. Therefore, if with centre E and radius EA, we describe a circle, it will touch the circle BCF at F, and A will be its highest point. Again, by Art. 41, AF is the line of quickest descent.

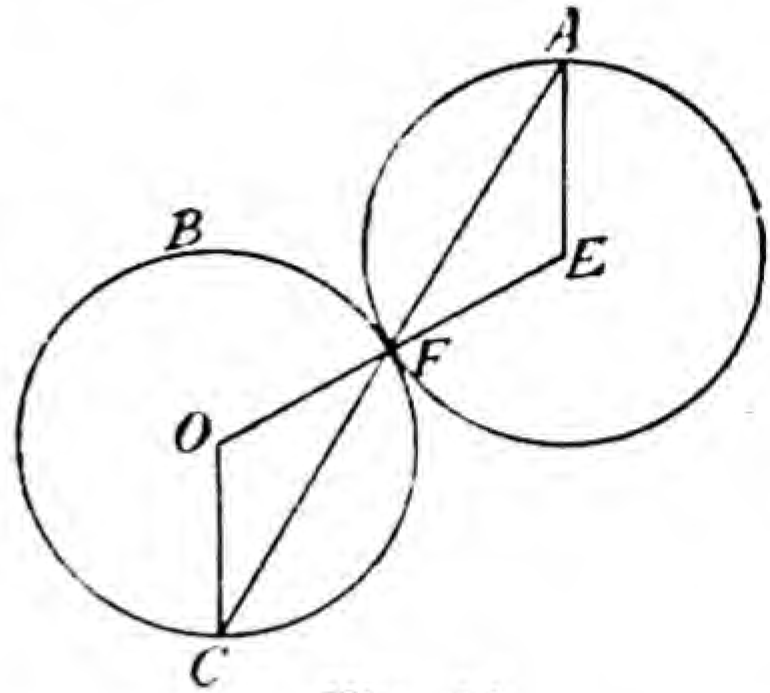


Fig. 33

47. The chord of quickest descent from a given point P to a curve in the same vertical plane is PQ, where Q is a point on the curve such that a circle, having P at its highest point touches the curve at Q.

For let a circle be drawn, having its highest point at P, to touch the given curve externally in Q. Take any other point Q_1, on the curve, and let PQ_1 meet the circle again in R.

Then, since PQ_1 is $>$PR,

the time down PQ_1 is $>$ time down PR.

But time down PR=Time down PQ (Art. 43) so that the time down PQ_1 is $>$time down PQ.

and Q_1 is any point on the given curve.

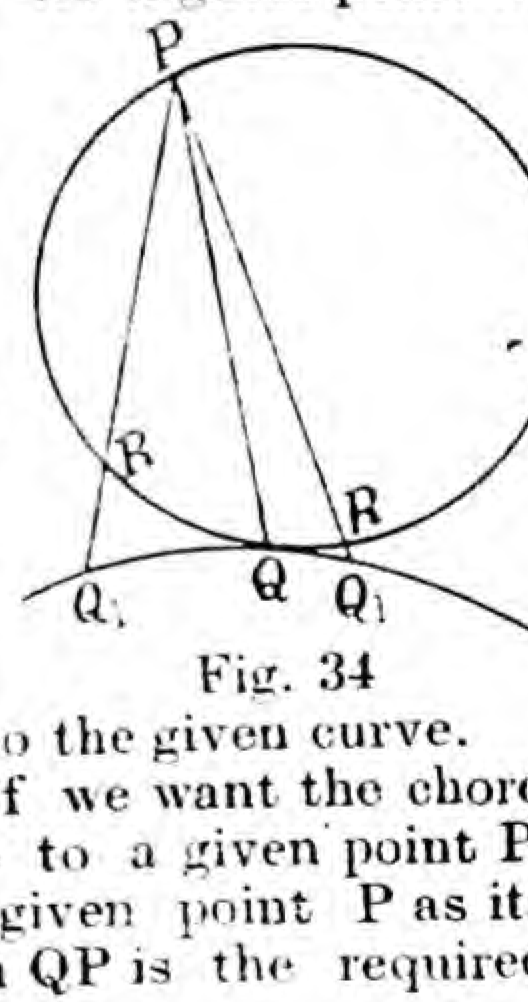


Fig. 34

Hence the time down PQ is less than that down any other straight line from P to the given curve.

Similarly, it may be shown that, if we want the chord of quickest descent from a given curve to a given point P, we must describe a circle having the given point P as its lowest point to touch the curve in Q; then QP is the required straight line.

Ex. Find the line of quickest descent—

(*a*) From a given point to a given straight line.

(*b*) From a given straight line to a given point.

(*c*) From a given straight line without a given circle to the circle.

(*d*) From a given circle to a given point without it.

(*e*) From a given circle to a straight line without it.

(*f*) From a given point within a circle to the circle.

(*g*) From a given circle to a given point within it.

(*h*) From a given circle to another circle within it.

(*i*) From a given circle within the circle to the circle.

(*j*) From a given circle to another circle without it.

Ex. 1. A heavy particle slides from rest down a smooth inclined plane, 15 ft. long and 12 ft. high. What velocity will it possess, when it reaches the bottom, and how many seconds will be occupied in the descent ? How long would it take to fall vertically through a height of 12 ft. ?

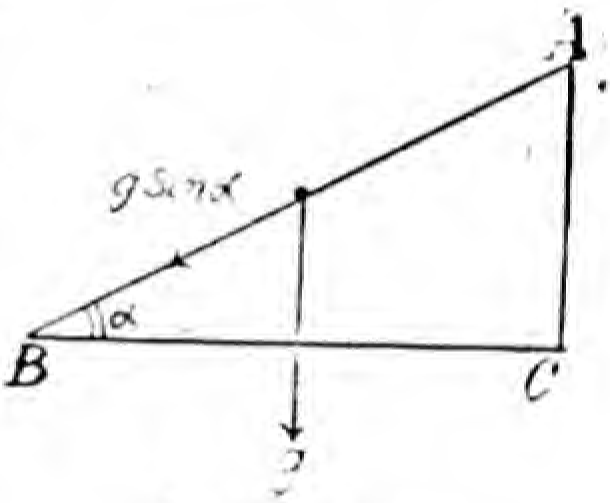


Fig. 35

Let AB be a smooth inclined plane.

$$AB = 15 \text{ ft} ; \ AC = 12 \text{ ft}.$$

The acceleration of the particle down the plane $= g \sin \alpha$

$$= \frac{32 \times 12}{15} \text{ ft./sec}^2.$$

The velocity at B is given by

$$v^2 = 2f.s$$

$$= \frac{2 \times 32 \times 12}{15} \times 15$$

$$\therefore \quad v = 16\sqrt{3} \text{ ft./sec.}$$

Time of descent

$$s = \tfrac{1}{2} ft^2.$$

$$15 = \tfrac{1}{2} \times \frac{32 \times 12}{15} t^2$$

or $$t = \frac{15}{8\sqrt{3}} \text{ sec.}$$

If it were to fall vertically, its acceleration would be 32 ft./sec². Hence the time taken to fall vertically a distance of 12 ft. is given by

$$12 = \tfrac{1}{2} \times 32t^2$$

$$\therefore \qquad t = \frac{\sqrt{3}}{2} \text{ secs.}$$

Ex. 2. A particle starting from an extremity of a horizontal diameter of a vertical circle of radius a slides down a smooth chord of the circle in t sec. Show that the inclination of the chord to the vertical is given by $4a \tan \theta = gt^2$.

Let BOB′ be the vertical diameter of the circle, and A the extremity of the horizontal diameter. Let AC be a chord through A making an angle θ with the vertical.

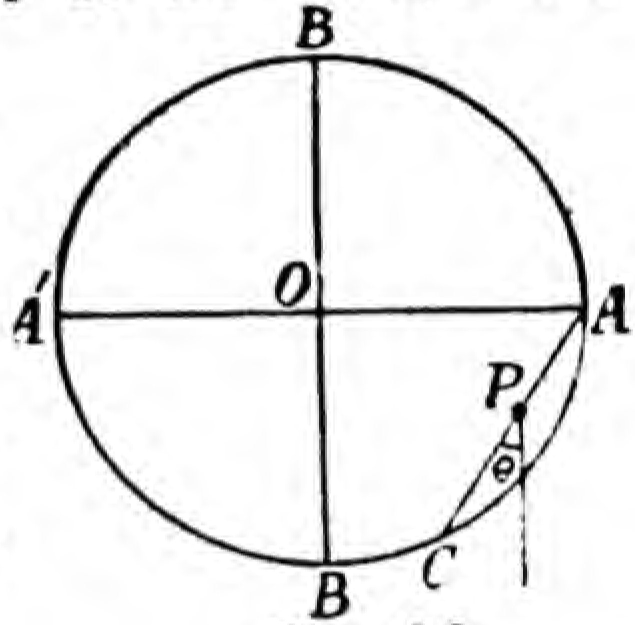


Fig. 36

Now AC, calculated from the $\triangle$OAC is

$$AC = 2a \cos OAC$$
$$= 2a \cos\left(\frac{\pi}{2} - \theta\right) = 2a \sin \theta.$$

The acceleration of P down the chord is $g \cos \theta$.

$$\therefore \qquad s = \tfrac{1}{2} ft^2.$$

$$2a \sin \theta = \tfrac{1}{2} g \cos \theta \, t^2$$
or $4a \tan \theta = gt^2$.

Examples V

1. A particle projected from the bottom up a smooth inclined plane, with a velocity of 24 ft. per sec. is just carried to the top, in 1 sec.; find the inclination of the plane to the horizon, and also the height of the plane.

2. The height of an inclined plane is $\frac{3}{5}$ of its length. A body is projected up the plane from the bottom with a velocity of 50 ft. per second, and slides down again; find the distance travelled up the plane, and the time before the body arrives at the starting point.

3. A particle starts from rest at the top of a smooth inclined plane of height h and inclination α to the horizon, and is constrained to slide down the plane along a straight line making an angle θ with the line of the greatest slope. Show that the velocity of the particle on reaching the ground is the same in magnitude as if it had fallen vertically under gravity. Find also the time of descent to the horizontal plane.

4. Two heavy bodies descend the height and length respectively of a smooth inclined plane; show that the ratio of the times equals the ratio of spaces described and the velocities acquired are equal.

5. A plane is of length 288 feet and of height 64 feet; show how to divide it into three parts so that a particle at the top of the plane may describe the portions in equal times and find these times.

6. A number of rods meet in a point A and rings placed on them slide down the rods, starting simultaneously from A. Show that after a time t the rings are all on a sphere of radius $\frac{gt^2}{4}$.

7. If two vertical circles touch at their highest points, and a straight line be drawn through this point cutting the circles, show that the time down the part between the circumferences is constant.

8. If a circle be placed in a vertical plane, determine that chord passing through its lowest point down which a body must fall so that it may acquire the greatest horizontal velocity.

9. AB is the horizontal diameter of a vertical circle, C the lowest point of the circle, P a point on the circumference. If t_1, t_2, t_3, the times of falling from rest down PA, PB, PC be such that $t_1^2+t_2^2=4t_3^2$, find the angle PAB.

10. Determine a point in the hypotenuse of a right-angled triangle, having its base horizontal from which the time of a particle's descent down an inclined plane to the right angle is least.

11. In a vertical circle two chords are drawn from the extremity of a horizontal radius subtending arcs θ and 2θ; if the time down the chord of 2θ is n times that down the chord of θ, show that $\sec\theta=n^2-1$.

12. Find the line of quickest descent of a particle from a given vertical circle to a point within it.

13. Two circles lying in the same plane touch each other externally and have their line of centres vertical; show that the time of descent of a particle, from the upper circle to the lower, along any smooth straight line passing through the point of contact is the same.

Miscellaneous Examples

1. A point at a certain instant has velocity 20 ft./sec. and it goes twice as far in the third and fourth seconds together as it does in the first two seconds of its motion; find the acceleration.

2. A point has $-m$ ft./sec². acceleration, and initially it has mn ft./sec².; show that after $(n+k)$ seconds it returns to the same point which it was passing at the end of the first $(n-k)$ seconds of its motion, with the same velocity.

3. A point having an acceleration a, passes over h feet in a certain interval, with an average velocity u, and its velocity is increased during the interval by v; prove that $uv=ah$.

4. A point moving with a constant acceleration a ft./sec²., passed over twice as many feet in a certain interval t seconds as it did in the immediately preceeding interval of t seconds, show that its velocity at the beginning of the first interval was $\frac{1}{2}t.a$ ft./sec.

5. A and B are two points in the same vertical line. From B, the lower of the two points, a heavy particle is projected vertically upwards with a velocity which will just carry it to A, and at the same time a heavy particle is just dropped from A. Show that when the particles meet their velocities will be equal and opposite, and the spaces passed over by the particles will be as 3 : 1.

6. A heavy body was observed to fall through 500 feet in 5 seconds; how far and for how long had it fallen before it was observed? (Assume that it started from rest.)

7. A stone dropped into a well reaches the water with a velocity of 80 ft./sec. and the sound of its striking the water is heard 27/12 seconds after it is let fall. Find from these data the velocity of sound in air.

8. Prove that the line of quickest descent from a given point to a given curve bisects the angle between the vertical and the normal to the curve at the point where it meets the curve.

9. A particle is projected upwards from the ground with a certain velocity, and after reaching a height 576 ft., it takes 5 seconds to return to this height; find the velocity of projection.

10. If the distance between O and the edge of the table is h feet, and a particle is projected from O with a velocity u ft./sec. while it is accelerated towards the edge with a ft./sec^2. and when it strikes the edge it is reflected with a velocity which is e times the velocity of impact, find how far it will move from the edge before losing its velocity, the acceleration towards the edge always continuing.

11. A tram-car starts from rest and accelerates uniformly for 8 seconds to a speed of 10 miles per hour. It then runs at a constant speed, and finally is brought to rest in 40 feet with a constant retardation. The total distance passed over is 250 yds Find the value of acceleration, the retardation, and the total time taken.

12. A ball hits a wall with a velocity 30 ft./sec., and after being in contact with the wall for 0·2 seconds it rebounds with a velocity of 10 ft./sec. At what rate has its velocity changed while in contact with the wall.

13. A body starting with initial velocity and moving with uniform acceleration acquires a velocity of 20 ft./sec. after moving through 10 ft. and a velocity of 30 ft./sec. after moving through a further 15 ft. When and where will its velocity be 40 ft./sec.?

14. A particle starts from O along a straight line with a uniform velocity of 4 ft./sec. After 3 seconds another particle starts from a point 8 feet from O in the same direction with a velocity of 8 ft./sec. and with an acceleration equal to 6 ft./sec^2. Find when and where will it overtake the first.

15. A point moves with uniform acceleration. If v_1, v_2, and v_3 be the average velocities in three successive intervals of time t_1, t_2 and t_3, show that

$$\frac{v_1-v_2}{v_2-v_3}=\frac{t_1+t_2}{t_2+t_3}.$$

16. An engine driver puts on his brakes and shuts off steam when he is running at full speed; in the first second afterwards the train travels 87 feet and in the next 85 feet. Find the original speed of the train, the time that elapses before it comes to rest and the distance it travels in this interval, assuming the brake to cause a constant retardation.

Find also the time the train will take, if it be 96 yards long, to pass a spectator standing at a point 484 yards ahead of train at the instant when the brake was applied.

17. A stone is let fall from a height of 240 ft., and at the same time another stone is thrown upwards to meet it. With what velocity must the second stone be projected in order that the stones may meet at a height of 96 feet?

18. A stone is thrown vertically upwards with such a velocity as will just take it to the level of a tower 100 ft. high. Two seconds later another stone is thrown up from the same place with the same velocity. Find when and where the two stones will meet.

19. A stone dropped into a mine is heard to strike the bottom in 79/16 sec. Find the depth of the mine, the velocity of sound being 1100 ft./sec.

20. A juggler keeps three balls going with one hand, so that at any instant two are in the air and one in his hand. Find the time during which a ball stays in his hand if each ball rises to a height (*i*) 16 ft. (*ii*) 8 ft.

21. A particle projected vertically upwards takes t secs. to reach a height 'h' feet. If t_1 secs. is the time from this point to the ground again, prove that $h=\frac{1}{2}gtt_1$ and that the maximum height is $\frac{g(t+t_1)^2}{8}$.

22. A, B, C, D are points in a vertical line the length AB=BC=CD. If a body falls from rest at A prove that the time of describing AB, BC, CD are

$$1 : \sqrt{2}-1 : \sqrt{3}-\sqrt{2}.$$

23. Two stopping points of a tram-car are 440 yds. apart, the maximum speed of the car is 20 m.p.h. and it covers the distance between the stops in 75 secs. If both acceleration and retardation be uniform, and the latter twice as great as the former find the value of each of them and also how far the car runs at its maximum speed.

24. A cruiser at A sights a friendly gunboat at B. S.-W. of it and 15 sea-miles away. If the gunboat is doing 18 knots due North and the cruiser can steam 22 knots find what course the cruiser should take to meet the gunboat as quickly as possible, and find the time taken.

25. A body starts from rest and moves in a st. line with uniform acceleration till its velocity is 20 ft./sec. It then moves with this velocity for some time and is finally brought to rest by a uniform retardation. The time taken during the accelerated motion is twice that

during the retarded motion. If the distance covered be 130 feet and the time taken be 8 secs, find the value of the acceleration, the retardation and the three periods of motion.

26. A body travels a distance s in t secs. It starts from rest and ends at rest. In the first part of the journey it moves with a constant acceleration f and in the second part with constant retardation r. Show that

$$t=\sqrt{2s\left(\frac{1}{f}+\frac{1}{r}\right)}.$$

27. It is observed that a ball strikes a block of wood with a velocity v and penetrates m feet; prove that in passing through a board n feet thick ($n < m$), resistance being uniform and same as before, it would lose the velocity

$$v\left\{1-\left(\sqrt{\frac{m-n}{m}}\right)\right\}.$$

28. Prove that the distances travelled over in successive intervals by a point starting from rest and moving with a constant acceleration are proportional to the series 1, 3, 5, 7..............

29. A body begins to slide down a smooth inclined plane from rest at the top, and at the same time another body is projected upwards from the foot of the plane with such a velocity that they meet half way up the plane. Find the velocity of projection and determine the velocity of each when they meet.

30. The side BC of a $\triangle$ ABC is vertical; show that if the times of falling down the two sides BA, AC be equal the triangle must be isosceles or right-angled.

31. A stone thrown vertically upwards passes a certain point t_1 seconds after projection, after a further t_2 seconds strikes the ground again. Prove that the height of this point above the point of projection is $16t_1t_2$ ft. Prove also that the stone passes a point midway between this point and the point of projection with velocity $16\sqrt{t_1^2+t_2^2}$ ft./sec.

32. Two men parting at a street corner walk off down different streets at steady speeds. Show that the line from one to other is always parallel to their resultant relative velocity.

Show that if they interchange directions without altering their speeds, their relative velocity will be unchanged in magnitude, but changed in direction, unless their speeds are equal.

33. A stone is dropped from the top of a tower into a well at its foot, reaching the water with a velocity 128 ft/sec. The sound of splash is heard $4\frac{8}{35}$ seconds after the stone was dropped. Find the velocity of sound.

34. A man stands on a platform which is ascending with a uniform acceleration of 6 ft./sec^2 and at the end of 4 seconds drops a stone. Find the velocity of the stone after three more seconds.

35. A stone is thrown vertically upwards with a velocity of 160 ft./sec. Another is dropped down a well at the instant the first is within 20 feet of your hand on its return journey. At what distance below your hand the two stones meet ?

36. A stone is dropped from the top of a cliff. The ratio of its velocities at the foot of the cliff and 112 ft. above the foot is 4 : 3. Find the height of the cliff.

37. Three planes are inclined at angles of 30°, 45°, 60° respectively. Find the distance a body must slide down each plane in order to acquire a velocity of 10 cms./sec,

38. A man in a lift, which is rising with a uniform acceleration 'f', throws a ball vertically upwards with a velocity of v ft./sec. relative to the lift, and after t secs. he overtakes it.

Prove $$f+g=\frac{2v}{t}.$$

39. A ship is sailing due N. at knots and observes another ship 4 sea-miles away at a bearing 55° W. of N. The latter ship is sailing at 14 knots due east. Find the time that elapses before the ships are nearest to one another and the distance in sea-miles which separates them. (1 Knot=6080 ft. per hour.)

40. Show that the time that a particle takes to slide down a chord of a vertical circle, starting from one end of a horizontal diameter, varies as the square root of the tangent of the inclination of the chord to the vertical.

CHAPTER V

THE LAWS OF MOTION

48. This chapter is devoted to the study of the Laws of Motion enunciated by Newton in his *Principia Mathematica*, and therefore it is necessary to discuss certain fundamental concepts introduced by him. Our aim is to give a critical appreciation of the Laws, keeping in view the spirit of his age. Newton was not the first Mathematician who was confronted with problems relating to moving bodies. He had Galileo (1564—1642), Christianus Hugenius (1629—1695), Kepler (1571—1630) as his able predecessors. Galileo is the father of Dynamics. He investigated the Laws of falling bodies, the Law of Inertia, and for the first time introduced the idea of acceleration and "efficacy" (force). Hugenius or (Huygens) made a number of important contributions to the subject: [The theory of the centre of oscillation; invention of pendulum clock, determination of the acceleration of gravity g, by pendulum observations; and certain theorems regarding centrifugal force.] Then there was Kepler who had deduced from the observations of Tycho Brahe (a Swedish astronomer) and his own, three empirical laws of motion of the planets about the sun.

We now discuss in some detail the achievements of Newton as they bear upon the Principles of Mechanics. Newton extended the subject along the following lines:—

(1) The generalisation of the idea of force.

(2) The introduction of the concept of mass.

(3) The distinct and general formulation of the principle of the parallelogram of forces.

(4) The statement of the Law of action and reaction.

Newton has expressed emphatically that he was not concerned with hypotheses as to the causes of phenomena, but has simply to do with the investigation and transformed statement of actual facts, a direction of thought that was distinctly and

tersely uttered in his words "*hypotheses non fingo*", "I do not frame hypotheses." This sums up his attitude.

49. Definitions. The **Mass** is the quantity of matter in the body and is measured by the product of its volume and density.

The **Density** of a uniform body is the mass of a unit volume of the body ; so that if m is the mass of volume v of a body whose density is ρ, then

$$m = v\rho.$$

It is obvious from these definitions that mass has been defined in terms of density and that density has again been defined in terms of mass. Moreover this definition does not take any cognizance of the inertia of matter. A new definition has been given by Ernst Mach.

Force is that which changes or tends to change, the state of rest or uniform motion of a body.

50. As a result of experimental investigation Newton felt distinctly that in every body there was inherent a property whereby the amount of its motion was determined. He called it, as we still do, mass ; but he did not succeed in correctly stating this perception. Numerous examples can be given in support of this view. If we take two pieces of different size of the same substance and place them on a smooth table, then the same effort on our part will be able to move the smaller body with more ease than the bigger one. Again, if we take two pieces of the same size and shape of two different substances and place them on a smooth table, the effect produced on the two by the same effort on our part will be different.

There is a slight pitfall in the above definition of force and it is probable that the enthusiastic beginner may be misled by it. The force defined in this way, as Newton expressly states, has nothing to do with the "unknown causes" producing motion. That which in mechanics is called force is not a something that lies latent in the natural processes, but a measurable actual circumstance of motion. It is more of the nature *of an explanation than a cause.* It is in terms of force that the motion becomes measurable and intelligible.

51. The British unit of mass is called the *Imperial Pound*, and consists of a lump of platinum kept in Westminster, of which there are several accurate copies kept in other places.

The French unit of mass is called a *gramme*, and is the one-thousandth part of a certain quantity of platinum kept in Paris. The gramme is defined as the mass of a cubic-centimetre of pure water at a temperature of 4° centigrade.

One Pound = about 453·6 grammes.

52. The **Weight** of a body is the force with which the earth attracts the body.

It is an observed fact that every particle of matter in nature attracts every other particle with a force, which varies directly as the product of the masses of the quantities, and inversely as the square of the distance between them. ($F = G.\frac{mm'}{d^2}$, where G is a certain universal constant of gravitation, and its value is $6{\cdot}66 \times 10^{-8}$ in C.G.S. system and $1{\cdot}05 \times 10^{-9}$ in F.P.S. system). Newton has shown that a sphere attracts a particle on, or outside, its surface with a force which varies inversely as the square of the distance of the particle from the centre of the sphere. (He took twenty years to solve this problem.) The earth is not a perfect sphere and different points of its surface are at different distances from its centre; hence the attraction of earth for a given body is different at different points, and therefore its weight varies at different places on the earth's surface.

53. Momentum. The momentum of a body is the product of its mass and velocity, and it has the same direction and sense as the velocity of the body. It is therefore a vector quantity.

A body of unit mass moving with a unit velocity is said to possess unit momentum. In F.P.S. system the unit of momentum is ft./lbs. per second. In C.G.S. system the unit is cms./gms. per second.

The momentum of a body of mass m (lbs) and velocity v (ft. sec.) $= mv$ (ft. sec. units of momentum) in the direction of velocity.

54. We give below the three Laws of Motion, as enunciated by Newton in his *Principia* published in the year 1686 :

Law I. *Every body preserves in its state of rest or of uniform motion in a straight line, except in so far as it is compelled to change that state by impressed forces.*

Law II. *The rate of change of momentum is proportional to the impressed force, and takes place in the direction of the straight line in which the force acts.*

Law III. *To every action there is an equal and opposite reaction.*

55. Criticism of the Laws. These laws and the aforesaid definitions form the very basis of Dynamics. Although the laws cannot be proved experimentally or otherwise, yet mathematicians for about three hundred years have not challenge their validity. There is surely something in the very statement of these laws which deludes human intelligence ; and, moreover, the results obtained and the predictions made, in the realm of Astronomy agree so well with the observed facts that no body ever dreamt of doubting the truth of these statements. There were, however, certain phenomena of a rather delicate nature which could not be explained on the hypothesis of gravitation and on the basis of these laws. The shift of 43 sconds per century in the case of perihelion of the planet Mercury ; the deflection of a ray of light when it passes in the neighbourhood of Sun ; and the change in the period of vibrations of an atom when the latter is found in different places ; and the change in mass that takes place in the case of an electron moving at a terrific speed, are some of the facts which baffled human understanding. It required the genius of Ernst Mach and Albert Einstein to point out that there was some fundamental error in our method of approach. The fundamental quantities of the classical Mechanics, the mass, the length and the time as supposed by the classical thinkers and even Newton himself were not independent of each other. There was nothing universal and absolute about these concepts. They were conditioned by the physical state of the observer measuring them. The Newtonian ideas respecting space and time were erroneous and needed a thorough revision. The Special and the General theories of Relativity propounded by

Albert Einstein in the year 1905, accomplish this task in a wonderful way. This, however, does not lessen the importance of these laws so far as their utility is concerned ; it simply robs them of their universality.

56. Law I. The first part of the Law enunciates the Principle of Inertia. It is also called the Law of Inertia. If, for the time, we exclude the second part of the first law of motion, and approach it with no notion of force, then we might say this : There is a kind of motion which is to be distinguished for its simplicity, the implication being that this kind is a preferred one and would be universal were it not for the intervention of some outside influence which prevents it from always being attained. In other words, there is an innate tendency in every material body, when *left to itself*, to perpetually remain in its own state of rest or of uniform motion in a straight line.

The state of rest of a body means that it occupies the same position relative to the surrounding objects. According to this law, the body never leaves that position *of its own accord*.

The state of uniform motion in a straight line means that the body traverses equal distances of a homogeneous space in equal times, however small the interval of time may be, provided there is no *outside influence*.

Granting our ignorance of force, the law becomes a pious statement of a possible state of motion which may sometimes be realized in nature and sometimes not.

The second part of the law furnishes us with a definition of force. It does not say what happens to a particle when a force acts on it ; rather it purports to say that a force is anything which acts to change the state of the particle from rest or uniform motion in a straight line. This would mean that, whenever we note a particle which is not in this ideal state, we are to say that it is acted upon by a force. The law may be looked upon as a kind of qualitative introduction to the second law of motion. One might reasonably expect that this law gives us the meaning of *zero force*. No force acts in the ideal state indicated. A little thinking shows that this leads to certain apparent perplexities. For example, a book resting on

a table might then be taken as an illustration of zero force. This contradicts the common sense feeling that the book actually exerts a force on the table. The difficulty can be cleared up at once by deciding that the state of rest or equilibrium involves the action of more than one force ; we meet here the notion of force pairs and resultants. A body at rest or in uniform motion is, therefore, acted upon by a number of forces which are in equilibrium.

57. Experimental Evidence. It has already been said that the truth of this proposition cannot be deduced in a strict manner from observation and experiment. The Law being an ideal statement is never realized in the world of everyday experience. Still under certain conditions we come across cases which at least give some ground for believing in the law.

(*i*) If a man be riding on a horse which is galloping at a fairly rapid pace and the horse suddenly stops, the rider is in danger of being thrown over the horse's head.

(*ii*) If a man stands upright in a railway carriage, then, so long as the motion of the train is uniform and in a straight line, he will not feel that he is being pushed forward in any way. But if the train suddenly stops, the man will fall forwards owing to his tendency to go on moving.

(*iii*) As an instance in which force is required to change motion, consider a stone whirled rapidly round and round at the end of a string. The stone describes a circle, not a straight line ; hence, according to the law the stone is being acted upon by some force which constantly changes its direction of motion. We find that unless we hold the end of the string firmly and exert a pull on it, the stone will fly right off. In fact, if it be whirled sufficiently rapidly, the force required to continually change its direction may become great enough to break the string, and the stone will then fly in a straight line.

(*iv*) If a smooth coin be placed on a smooth card placed over the rim of a tumbler and if the card be given a brisk motion along its surface, the coin at once falls into the tumbler, showing its tendency to remain at rest.

58. Law II. From this law we derive a quantitative measure of force.

Let m be the mass of a body, and f the acceleration produced in it by the action of a force whose measure is P.

Then by the second law of motion,

$P \propto$ rate of change of momentum.

$\propto$ rate of change of mv.

$\propto m \times$ rate of change of v (if mass is an invariant).

$$\propto m.\frac{v-u}{t}$$

where u is the initial velocity, and v the velocity at the end of time t.

$\therefore \quad P \propto m.f \qquad [\text{as, } v=u+ft]$

$\therefore \quad P=\lambda\, mf,$ where λ is some constant.

Now let the unit of force be so chosen that it may produce in unit mass, the unit acceleration.

Hence, when $m=1$, $f=1$, we have $P=1$

$\therefore \quad \lambda=1$

The unit of force being thus chosen, the expression for force becomes,

$$P=m.f.$$

59. Units. In F.P.S. system, m is measured in pounds, f in ft./sec^2. Substituting in the above equation, we get, P in Poundals.

In C.G.S. system, m is measured in grammes, f in cms./sec^2., the corresponding unit of P is called a Dyne. A Poundal and a Dyne are called Absolute or Dynamical units of force.

The force which acting on a mass of one pound produces an acceleration of one foot per second per second is called a ***Poundal.***

The force which acting on a mass of one gramme produces an acceleration of one centimetre per second per second is ***called a Dyne.***

60. Comparison of forces. If two forces P and P′ acting on a mass m produce accelerations f and f' respectively, then

$$P = m.f.$$
$$P' = mf'$$

$$\therefore \quad \frac{P}{P'} = \frac{f}{f'} = \frac{ft}{f't},$$ t being the time during which each force acts on m.

$$\therefore \quad \frac{P}{P'} = \frac{\text{change in vel. produced by P during time } t}{\text{change in vel. produced by P}' \text{ during time } t}$$

If $P = P'$, then they produce equal changes in velocity in the same mass in equal times.

61. Comparison of masses. If a force P produces in the masses m and m', accelerations f and f' during the time t, we have

$$P = mf = m'f'$$

$$\therefore \quad \frac{m}{m'} = \frac{f'}{f} = \frac{f't}{ft} = \frac{\text{change in the vel. of } m'}{\text{change in the vel. of } m}.$$

If $m = m'$, then a given force acting upon them for equal intervals of time produces in them equal changes in velocity.

62. Relation between mass and weight. Let W be the weight of a body of mass m. Since weight is defined as the force which the earth exerts on the body,

$W = m.g$, where g is the acceleration produced in the body due to the pull of the earth.

In F.P.S. system $g = 32$ ft./sec^2. approximately.

In C.G.S. system $g = 981$ cm./sec^2. approximately.

63. Gravitational units of force. The weight of a body of mass $m = mg$ absolute units of force

$\therefore$ The weight of a body of unit mass $= g$ absolute units of force

$\therefore$ The weight of 1 lb. $= g$ poundals.

The weight of unit mass is called the Gravitational unit of force.

The weight of one pound is called a pound-weight, which is often abbreviated into a pound. A pound, therefore, may mean either a mass of 1 lb. or a force equal to lb. wt.

64. Distinction between mass and weight. The mass of a body is an invariant quantity. It always remains the same as long as the body remains unchanged. The weight of a body is the force with which it is attracted to the centre of the earth and as the acceleration due to gravity is slightly different at different places of the earth's surface, the weight of a body will slightly change from place to place.

65. Physical Independence of forces. The principle of physical independence of forces follows from the second part of Newton's second law of motion. It states that every force acting on a body produces its effect in the same direction in which it is acting. The effect of that force remains unaltered whether some more forces are introduced in the system or taken away from it.

If a body be resting on a table and if a force F_1 displace it from A to B and if another force F_2 acting singly displace it from A to C, then the displacement of the body when both F_1 and F_2 are acting on it would be the resultant of the displacements AB and AC.

It is found that a ball falling from the hands of a passenger travelling in a moving train, falls exactly below the spot where it was left. This is due to the fact that the newly introduced force of gravity does not interfere with the motion of the ball in the direction of the motion of the train.

66. Parallelogram of forces. In Art. 32 if m be the mass of the particle whose accelerations f_1 and f_2 are represented in magnitude and direction by the lines AB and AC, then its resultant acceleration f_3 is represented in magnitude and direction by AD, the diagonal of the parallelogram ABCD.

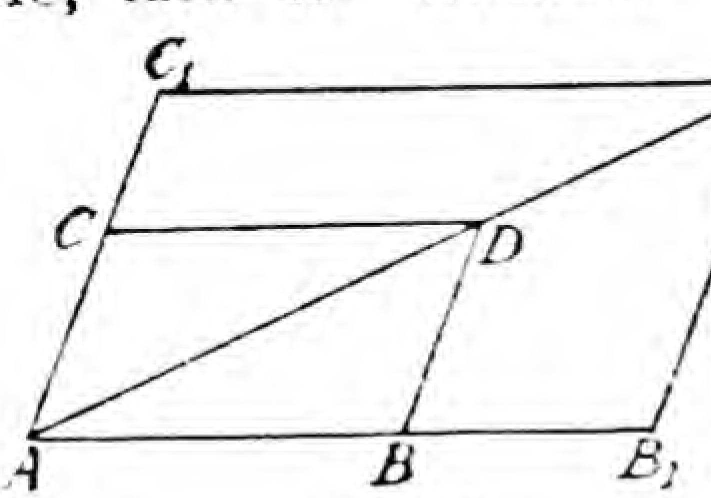


Fig. 37

Forces in the directions AB and AC, by the 2nd law, are mf_1 and mf_2. Let AB_1 and AC_1 represent these forces. The resultant of these forces, can be obtained by completing the parallelogram $AB_1D_1C_1$.

In the $\triangle$s ABD and AB_1D_1

$$\frac{AB_1}{AB}=\frac{B_1D_1}{BD}=\frac{m}{1}.$$

Therefore, A, D and D_1 are collinear. Hence AD_1 represents the force which produces the acceleration AD_1 and is therefore equivalent to the forces represented by AB_1 and AC_1.

Theorem. *If a particle be acted on by two forces represented in magnitude and direction by the two sides of a parallelogram drawn from a point, they are equivalent to a force represented in magnitude and direction by the diagonal of the parallelogram passing through that point.*

67. Law III. The most important achievement of Newton with respect to principles is the distinct and general formulation of the law of the equality of action and reaction, of pressure and counter-pressure. A body that presses or pulls another body is pressed or pulled in exactly the same degree by the other body. As the measure of force is the momentum generated in unit time, it consequently follows that bodies that act on each other communicate to each other in equal intervals of time equal and opposite quantities of motion (momenta). This leads to the law of Conservation of Momentum.

68. Experimental evidence. (1) If a book rests on a table, the book presses the table with a force equal and opposite to that which the table exerts on the book.

(2) *Pull or Tension*. A particle is tied at one end of an inextensible string the other end of which is tied to a fixed support. The particle due to the action of its weight exerts a pull on the support. This pull is communicated to the support through the string, which appears in the latter as Tension and is shown in Fig. 38 (*a*). If the support be sufficiently strong the particle hangs in equilibrium. The supports exert an upward force which also is communicated through the string to the particle. This is shown in Fig. 38 (*b*). For the equilibrium of the support, R = T Fig. 38 (*a*).

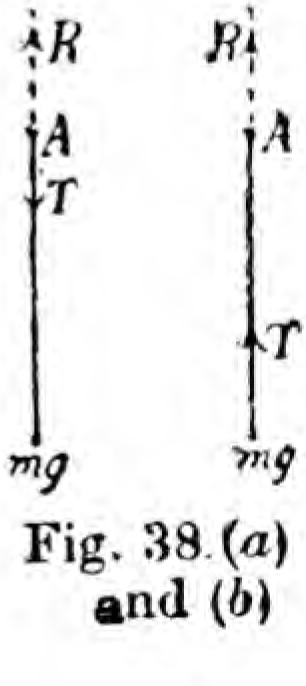


Fig. 38 (*a*) and (*b*)

(3) *Attraction and Rupulsion.* When two bodies act at a distance on each other then the action between them is called attraction or repulsion. There is an attractive force between the sun and the earth, which keeps the latter moving in an elliptic orbit about the sun. The amount of force by which the sun attracts the earth is equal and opposite to the attractive force of the earth on the sun. When similar poles of a magnet are placed near one another they repel each other and tend to separate away. The repulsions between them are equal in magnitude but opposite in direction.

Ex. 1. A force of 15 poundals acts upon a mass of 13 lbs. What velocity will it generate in 8 seconds ?

According to the 2nd law

$$P = mf$$

or

$$15 = 13f$$

or

$$f = \frac{15}{13} \text{ ft./sec}^2, \quad \text{is the acceleration}$$

acting on the mass. If the mass is at rest initially, the velocity after time t is given by

$$v = ft \qquad [\text{Putting } u = 0 \text{ In } v = u + ft]$$

or

$$v = \frac{15}{13} \times 8 = 9\frac{3}{13} \text{ ft./sec.}$$

Ex. 2. Find the magnitude of the force which acting on a mass of 10 cwt. for 10 seconds will generate in it a velocity of 3 miles per hour.

3 miles per hour are equivalent to $\frac{22}{5}$ ft./sec.

Since the body is at rest initially, the acceleration is given by

$$v = ft$$

$$\frac{22}{5} = f.\ 10$$

or

$$f = \frac{11}{25} \text{ ft./sec}^2.$$

Therefore the force P is given by

$$P = mf$$
$$= \frac{10 \times 112 \times 11}{25} \text{ poundals}$$

or

$$= \frac{10 \times 112 \times 11}{25 \times 32} \text{ lbs. wt.}$$
$$= 15 \frac{2}{5} \text{ lbs. wt.}$$

Ex. 3. A mass of 10 lbs. falls 10 feet from rest and is then brought to rest by penetrating 1 foot into some sand, find the average thrust of the sand on it.

The velocity acquired in falling through 10 feet under gravity from rest is given by,

$$v^2 = 2gh$$
$$= 2 \times 32 \times 10 \text{ ft./sec.}$$

This velocity becomes zero, when the mass has traversed 1 foot in the sand. Therefore the retardation is given by,

$$0 = 640 - 2f$$

or

$$f = 320 \text{ ft./sec}^2.$$

Therefore, the resistance of the sand P, is given by

$$P - 10g = mf$$

or

$$P \quad = 320 + 10 \times 320 \text{ poundals}$$
$$= \frac{110 \times 32}{32} \text{ lbs wt.}$$
$$= 110 \text{ lbs. wt.}$$

Ex. 4. A bullet of mass 1 oz. has a velocity of 500 ft./sec. when it strikes and remains embedded in a block of wood of mass 6 oz. moving in a perpendicular direction with a velocity of 500 ft. per sec. What is the final velocity and direction of motion of the block ?

Due to physical independence of the forces, the velocity of 500 ft./sec. remains unaffected after the impact of the bullet. The block, however, acquires a velocity perpendicular

to the velocity of 500 ft./sec. This velocity can be calculated by the law of the conservation of momentum. If v is velocity acquired by the block and the bullet after the impact, we have,

$$\frac{1}{16}\ 500 = \frac{7}{16} \times v$$

or
$$v = \frac{500}{7} \text{ ft./sec.}$$

Hence the resultant velocity of the block is

$$\sqrt{(500)^2 + \left(\frac{500}{7}\right)^2} = \frac{500}{7}\sqrt{50} \text{ ft./sec.}$$

making an angle $\theta = \tan^{-1}(\frac{1}{7})$ with the direction of its initial motion.

Ex. 5. A jet of water issues vertically at a speed of 30 feet per second from a nozzle of 0·1 square inch section. A ball weighing 1 lb. is balanced in the air by the impact of water on its underside. Show that the height of the ball above the level of the jet is 4·6 feet approximately.

Let v be the velocity of the jet at the height h, then,

$$v^2 = 30^2 - 2gh$$

If the ball balances in air at the height h, then,

The downward force due to the weight of the ball = The upward force supplied by the jet.

As soon as the jet strikes the ball at the height h, its velocity v is completely destroyed. Hence the momentum of the issuing water per second destroyed by impact, is the upward force. Therefore

$$1 \cdot g = (\text{mass of water issuing per second}) \times \text{velocity at the height } h$$

$$= \left(\frac{1}{1440} \times 62{\cdot}5 \times 30\right) v$$

or
$$v = \frac{48 \times 32}{62{\cdot}5} = 24{\cdot}576 \text{ ft./sec.}$$

$$\text{Therefore} \quad h = \frac{900}{64} - \frac{(24\,576)^2}{64}$$

$$= 4 \cdot 6 \text{ feet approximately.}$$

EXAMPLES VI

1. A man seated on a smooth seat in a railway carriage (with his back to the engine) tends to move when the train starts—in which direction? Why? Would he have any difficulty in keeping his seat when the train is going at full speed? Why?

2. If you jump off a moving train why do you tend to fall down on reaching the platform? In which direction are you likely to fall?

3. What force must be applied for one-tenth of a second to a mass of 10 tons in order to produce in it a velocity of 3840 ft. per minute? What would be the momentum of the mass so moving?

4. A force of 100 dynes acts for three seconds on a body of mass 25 grammes initially at rest, and then ceases to act. How far does the body travel in 4 secs. and in the 4th second?

5. Enunciate the 2nd Law of Motion and prove by means of it that if two bodies at rest receive two equal blows, the velocities produced are inversely proportional to the masses of the bodies.

6. A mass of 40 lbs. falls 10 feet from rest and then penetrates to a depth of 1 foot into the sand before coming to rest. Find the average thrust of the sand.

7. A horizontal force equal to the weight of 9 lbs. acts on a mass along a smooth horizontal plane; after moving through a space of 25 feet the mass has acquired a velocity of 10 feet per second; find its magnitude.

8. It was found when one foot was cut off from the muzzle of a gun firing a projectile of 100 lbs., the velocity of the projectile was altered from 1490 to 1330 feet per second. Show that the force exerted on the projectile by the powder-gas at the muzzle, when expanded in the bore, was about 315 tons weight.

9. A railway train whose mass is 100 tons moving at the rate of a mile a minute, is brought to rest in 10 secs. by the action of a uniform force. Find how far the train runs during the time for which the force is applied.

10. A mass m is acted on by a constant force of P lbs. wt. under which in t secs. it moves a distance x feet, and cquires a velocity v ft. per second. Show that

$$x = \frac{gt^2P}{2m} = \frac{v^2m}{2gP},$$

11. A bullet weighing 81 grammes and moving at the rate of 200 cms. per second is fired into a log of wood into which it penetrates 10 cms. If the bullet moving with the same velocity were fired into a similar piece of wood 5 cms thick, with what velocity would it emerge ? Find also the force of resistance, supposing it to be uniform.

12. A train of mass 120 tons is travelling with uniform speed, the resistance due to friction being 14 lbs. wt. per ton. If a portion of mass 20 tons is slipped, how much will the other portion have gained on it in 12 seconds, assuming the pull of the engine and the resistance per ton to be the same as before ?

CHAPTER VI

LAWS OF MOTION (*Continued*)

69. Motion of two particles connected by a string. *Two particles, of masses m_1 and m_2, are connected by a light inextensible string which passes over a small smooth fixed pulley. If m_1 be $> m_2$, find the resulting motion of the system, and the tension of the string.*

Let the tension of the string be T poundals ; the pulley being smooth, this will be same throughout the string.

Since the string is **inextensible**, the velocity of m_2 upwards must, throughout the motion, be the same as that of m_1 downwards.

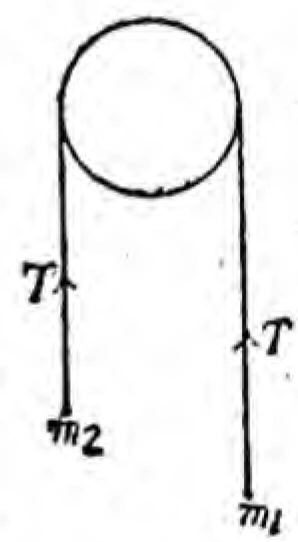


Fig. 39

Hence the accelerations of the two particles are equal in magnitude but opposite in direction. The acceleration of m_1 is directed downwards, while that of m_2 is directed upwards. Let the magnitude of the common acceleration be f ft./sec^2.

Then, by the 2nd Law of motion, the downward force on m_1 equals the product of its mass and acceleration ;

Hence $\quad m_1 g - \mathrm{T} = m_1 f \qquad \ldots(1)$

Similarly for the second particle, the upward force equals the product of its mass and acceleration.

$$\mathrm{T} - m_2 g = m_2 f \qquad \ldots(2)$$

Adding (1) and (2)

$$(m_1 - m_2)\, g = (m_1 + m_2) f$$

or

$$f = \frac{m_1 - m_2}{m_1 + m_2}\, g.$$

Substituting this value of f in (1), we get

$$\mathrm{T} = m_2(f + g) = \frac{2m_1 m_2}{m_1 + m_2}\, g \text{ poundals.}$$

Ex. Two weights W and W′ are connected by a light string passing over a light pulley. If the pulley moves upwards with an acceleration equal to that of gravity, show that the tension of the string is $\frac{4WW'}{W+W'}$.

If m_1 and m_2 be the masses of the two weights, we have

$$m_1 = \frac{W}{g} \text{ and } m_2 = \frac{W'}{g}.$$

Suppose f is the common acceleration of the two masses, if the pulley were at rest. Then due to the upward acceleration of the pulley, the acceleration of m_1 is $(f-g)$ vertically downwards, and that of m_2 is $(f+g)$ vertically upwards. If T is the tension of the string, we have

$$m_1 g - T = m_1 (f-g) \qquad \ldots(1)$$

and $$T - m_2 g = m_2 (f+g) \qquad \ldots(2)$$

Multiply (1) by m_2 and (2) by m_1 and subtract (2) from (1);
$2m_1 m_2 g - (m_1 + m_2) T = -2 m_1 m_2 g$

or $$T = \frac{4m_1 m_2}{m_1 + m_2} g$$
$$= \frac{4WW'}{W+W}.$$

70. *Two particles, of masses m_1 and m_2 are connected by a light inextensible string; m_2 is placed on a smooth horizontal table and the string passes over a light smooth pulley at the edge of the table, m_1 hanging freely. Find the resulting motion.*

Let T be the tension of the string in poundals.

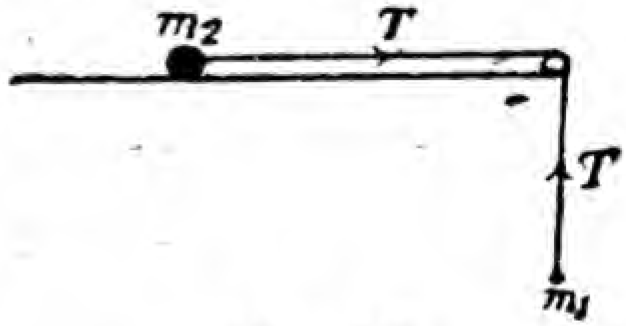


Fig. 40

The velocity and acceleration of m_2 along the table must be equal to the velocity and acceleration of m_1 in a vertical direction.

Let f be the common acceleration in feet per sec. per sec.

The force on m_1 is $m_1 g - T$, acting vertically downwards

$$\therefore \quad m_1 g - T = m_1 f \qquad \ldots(1)$$

The only horizontal force acting on m_2 is the tension T [for the weight of m_2 acts vertically downwards and is balanced by the reaction of the table]

$$\therefore \qquad T = m_2 f \qquad \ldots(2)$$

Adding (1) and (2), we have

$$m_1 g = (m_1 + m_2) f$$

$$\therefore \qquad f = \frac{m_1}{m_1 + m_2} g.$$

Substituting in (2), we get

$$T = \frac{m_1 m_2}{m_1 + m_2} g \text{ poundals.}$$

71. *Two masses, m_1 and m_2, are connected by a string; m_2 is placed on a smooth plane inclined at an angle α to the horizon, and the string, after passing over a small smooth pulley at the top of the plane, supports m_1, which hangs vertically; if m_1 descends, find the resulting motion.*

Let the tension of the string be T poundals. The velocity and acceleration of m_2 up the plane are equal to the velocity and acceleration of m_1 vertically downwards.

Let f be the common acceleration. The equation of motion of m_1 is,

$$m_1 g - T = m_1 f \qquad \ldots(1)$$

Forces on m_2 are: its weight $m_2 g$ acting vertically downwards, the reaction R perpendicular to the inclined plane, and the tension T of the string urging it to move up the plane.

Resolving $m_2 g$ along and perpendicular to the inclined plane, we note that the resolved part of $m_2 g$ perpendicular to the inclined plane is completely balanced by the normal reaction R, and the force $(T - m_2 g \sin \alpha)$ up the plane produces acceleration f.

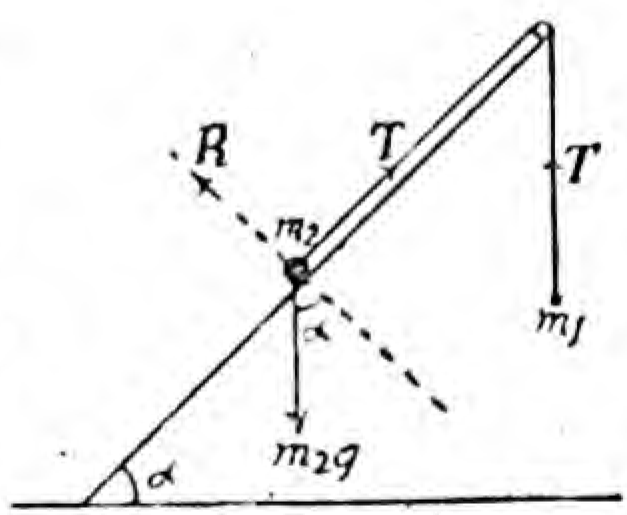


Fig. 41

Therefore, we have

$$R = m_2 g \cos \alpha \qquad \ldots(2)$$

$$T - m_2 g \sin \alpha = m_2 f \qquad \ldots(3)$$

Adding (1) and (3), we get

$$f = \frac{m_1 - m_2 \sin \alpha}{m_1 + m_2} g.$$

Substituting the value of f in (1), we have

$$T = m_1(g - f) = m_1 g \left[1 - \frac{m_1 - m_2 \sin \alpha}{m_1 + m_2} \right]$$

$$= \frac{m_1 m_2 (1 + \sin \alpha)}{m_1 + m_2} g$$

giving the tension of the string.

72. Motion of connected bodies on two inclined planes. *Two masses, m_1 and m_2 ($m_1 > m_2$) are connected by an inextensible but flexible string and placed over two smooth inclined planes making angles α and β with the horizontal, having a common vertex C, with the string passing over a smooth pulley at C. To find the motion.*

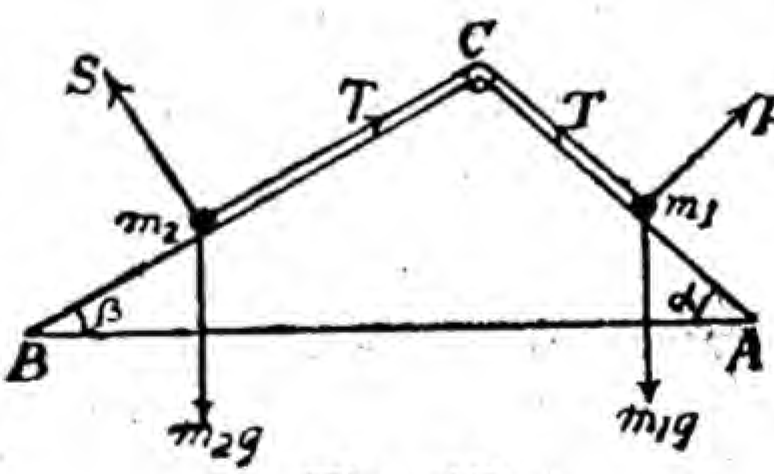


Fig. 42

Let f = common acceleration
T = tension of string in poundals.
R = normal reaction of the plane on m_1
S = normal reaction of the plane on m_2.

Since the particle m_1 moves down the plane with acceleration f we have,

$$R = m_1 g \cos \alpha \qquad \ldots(1)$$

and $$m_1 g \sin \alpha - T = m_1 f \qquad \ldots(2)$$

The particle m_2 moves up the plane with acceleration f,

$\therefore$ $$S = m_2 g \cos \beta \qquad \ldots(3)$$

$$T - m_2 g \sin \beta = m_2 f \qquad \ldots(4)$$

Adding (2) and (4), we get

$$g\,(m_1 \sin \alpha - m_2 \sin \beta) = (m_1 + m_2)\,f$$

or

$$f = \frac{m_1 \sin \alpha - m_2 \sin \beta}{m_1 + m_2}\,g$$

Substituting in (2), we get

$$T = m_1(g \sin \alpha - f)$$

$$= m_1 g \left(\sin \alpha - \frac{m_1 \sin \alpha - m_2 \sin \beta}{m_1 + m_2} \right)$$

$$= \frac{m_1 m_2 (\sin \alpha + \sin \beta)}{m_1 + m_2}\,g.$$

Ex. 1. A string is attached at a fixed point A, passes vertically downward and around a movable smooth pulley B of weight 50 lbs., then vertically upward and over a fixed pulley C, and then is attached to a weight of 35 pounds which hangs vertically. The 35 lbs. weight is given a downward velocity of 20 ft./sec. Find the tension in the string and the velocity of 35 lb. weight after it has moved 10 feet.

Let s be the displacement in feet of the 35-pound body in t seconds, and let f be its acceleration vertically downwards. Then, the acceleration of the 50-pound body is equal to $\frac{f}{2}$ vertically upwards.

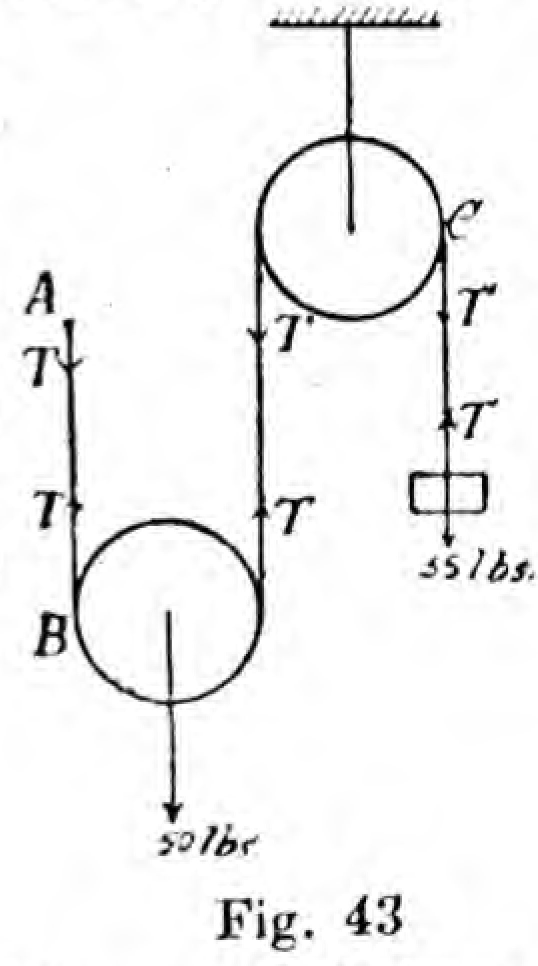


Fig. 43

Equation of motion of 35 lbs. body is

$$35\,g - T = 35\,f \qquad \ldots(1)$$

Equation of motion of 50 lbs. body is

$$T - 50g = 50 \times \frac{f}{2} \qquad \ldots(2)$$

Multiply (1) by 2 and add (2) to it

$$20\,g = 95\,f$$

$$\therefore \qquad f = \frac{4}{19}g \text{ ft./sec.}^2$$

And $\qquad T = 27{\cdot}6$ lbs. wt.

Now, initial velocity of 35-pound weight is 20 ft./sec ; and its acceleration is $\frac{4}{19} \times 32$ ft./sec². Hence, its velocity

after it has moved 10 feet, is given by

$$v^2=u^2+2fs$$

$$v^2=20^2+\frac{2\times4\times32\times10}{19}$$

$$\therefore \qquad v=23\cdot2 \text{ ft./sec.}$$

Ex. 2. A and B are masses of 6 oz. and 3 oz. respectively resting on two smooth tables, placed with their edges parallel. They are connected by a fine string which hangs between the tables with its hanging parts vertical and carries in its loop a smooth pulley C of mass 4 oz. The string lies in a vertical plane and crosses the edges of the table at right angles to the edges.

Find the tension in the string, (*i*) when A and B are held fast, (*ii*) when B is held but A moves, (*iii*) when A and B both move ; and show that in the three cases tensions are in the ratio 21 : 18 : 14.

(*i*) Let T be the tension in the string when A and B are held fast. The pulley C is at rest and we get, on considering its equilibrium,

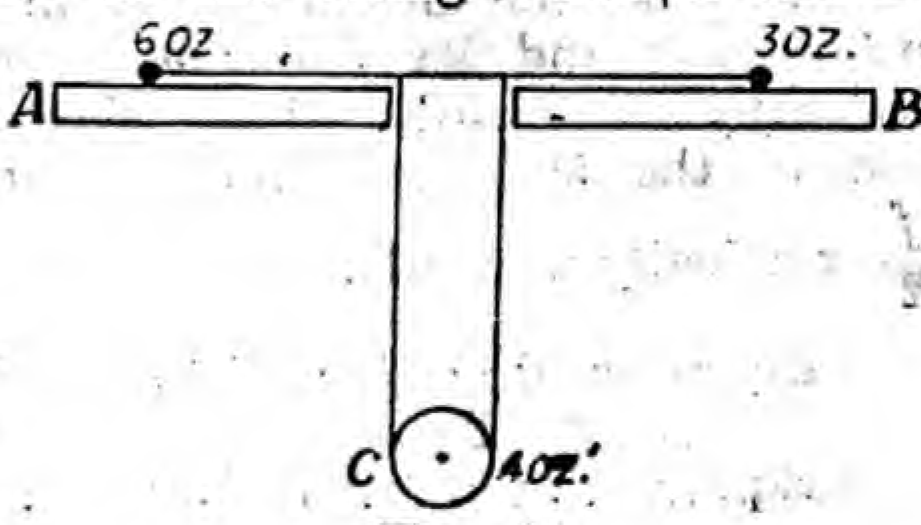


Fig. 44

$$2T-\frac{4}{16}g=0$$

or $$T=\tfrac{1}{8} \text{ lb. wt.} \quad ...(1)$$

(*ii*) When B is held let A move a distance x along the table in t seconds ; then C will move a distance $\frac{x}{2}$ in the same time. Let f be the acceleration of A, then

$$x=\tfrac{1}{2}ft^2 \qquad ...(2)$$

If f_1 is the acceleration of the pulley C, we have

$$\frac{x}{2}=\tfrac{1}{2}f_1t^2 \qquad ...(1)$$

or $$f_1=\frac{x}{t}=\frac{f}{2}. \qquad ...(2)$$

For the motion of A, if T is the tension in the string, we have

$$T=\tfrac{3}{8}f \qquad ..(3)$$

For the motion of C

$$\frac{1}{4}g-2T=\frac{1}{4}\ \frac{f}{2} \qquad ...(4)$$

From (3) and (4), we get

$$T=\frac{3}{28}\text{ lb. wt.} \qquad ...(5)$$

(*iii*) Let now A move a distance x and B a distance y in time t; then C moves a distance $\frac{1}{2}(x+y)$ in time t. If f_1 and f_2 are accelerations of A and B, the acceleration of C is $\frac{1}{2}(f_1+f_2)$. For the motion of A, B and C we have

$$\tfrac{3}{8}f_1=T$$

$$\tfrac{3}{16}f_2=T$$

and $$\tfrac{1}{4}g-2T=\tfrac{1}{4}\cdot\tfrac{1}{2}(f_1+f_2)$$

whence we get $$T=\tfrac{1}{12}\text{ lb. wt.} \qquad ...(6)$$

From (1), (5) and (6), we get the ratio of the tensions, as

$$\tfrac{1}{8}:\tfrac{3}{28}:\tfrac{1}{12}$$

or $$21:18:14.$$

Examples VII

1. Two particles, of masses 7 and 9 lbs. are connected by a light string passing over a smooth pulley. Find (1) their common acceleration, (2) the tension of the string, (3) the velocity at the end of 5 seconds, and (4) the distance described in 5 seconds.

2. Weights of 8 and 10lbs. are attached to the ends of a string which is placed over a smooth pulley. The system is given an initial velocity of 6 ft./sec. which causes the smaller weight to descend and the larger to rise. Find (*a*) the time until the weights reverse the direction of their motion; (*b*) the distance each body moves in that time.

3. Two particles of masses 10 pounds and 5 pounds are connected by a weightless, inextensible string, which is hung over a smooth pulley and let go. Two seconds after the motion starts the string is cut. After the string is cut, find how much farther the lighter weight continues to rise. Find the distance fallen by the heavier weight from the time the string is cut to the time the lighter weight ceases rising.

4. Two masses, each equal to m, are connected by a string passing over a smooth pulley; what mass must be taken from one and added to the other, so that the system may describe 200 feet in 5 seconds?

5. If ACB be a string passing over a smooth pulley C, and a weight of 5 lbs. be attached at A, a weight of 3 lbs. at B, and another of 3 lbs. between B and C, and if B be originally 11 feet from the ground, find the distance above B of the third weight in order that the latter may just reach the ground. Find also the time of motion.

6. A smooth pulley carrying a total load W hangs in a loop of a cord which passes over two fixed pulleys, and has weights P and Q freely suspended from its ends, each segment of the cord being vertical. Show that W will remain at rest or move with uniform velocity provided $\frac{1}{P}+\frac{1}{Q}=\frac{4}{W}$, there being no friction anywhere.

. A 2 lb. weight rests on the floor and is attached to a vertical string which passes over a smooth pulley. The other end of the string is attached to a 1 lb. weight which hangs 1 foot clear of the floor. The one lb. weight is raised vertically to a height h feet above the floor and then released. Find the value of h if the 1 lb. weight is just to touch the floor on its first descent

73. Motion of particles on rough surfaces. Earlier, we have had enough problems relating to motion of particles on surfaces which are perfectly smooth. Although, it is possible to diminish the roughness of a surface considerably by polishing it with suitable material, yet it cannot be made perfectly smooth. It is well known from experience that the action between two solid bodies in contact is not always merely a force perpendicular to the surface of separation. The simple fact that a body can rest on an inclined plane under no forces but its weight and the reaction of the plane shows that this reaction has a component along the plane which balances the component of weight down the plane. This component of reaction lies wholly in the tangent plane passing through the point of contact of the bodies and is called **Friction**.

Friction is a resisting force and is called into play only when it is necessary to prevent or oppose the relative motion of the point of contact. It cannot by itself produce motion of a body. It is a passive force. *The only function of a passive force is to preserve equilibrium if it can.*

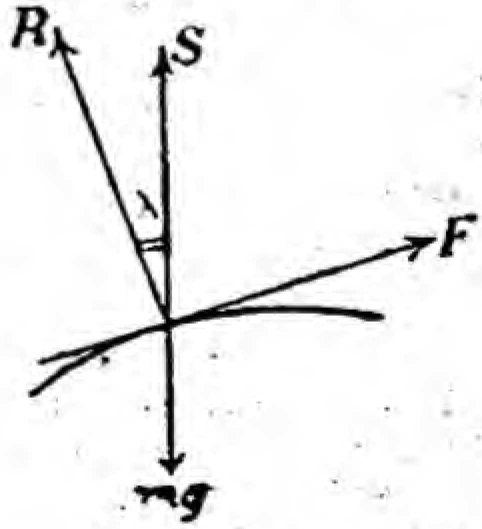


Fig. 46

74. Laws of friction. Friction may be of three kinds :—

Statical friction. When one body in contact with another is in equilibrium, the friction exerted is just sufficient to maintain equilibrium and is called *Statical friction.*

Limiting friction. When one body is just on the point of sliding on another, the friction exerted attains its *maximum value* and is called *Limiting friction.*

Dynamical friction. When motion ensues by one body moving or sliding on another, the friction exerted is called Dynamical friction.

75. Direction and Magnitude of Friction. When two bodies are in contact, the direction of friction on one of them at its point of contact is **opposite** to the direction in which the point of contact would move or in which it actually moves.

The magnitude of the limiting friction at the point of contact between two rough bodies bears a constant ratio to the normal reaction at that point. $\frac{F}{R}=\mu$, μ is called the coefficient of limiting friction. Also $\frac{F}{R}=\tan\lambda=\mu$, λ is called the angle of friction. λ is the angle which the resultant reaction S makes with the direction of R, the normal reaction.

When motion ensues by one body sliding over another, the magnitude of the friction is independent of the velocity of the point of contact, but the ratio of the friction to the normal reaction is slightly less when the body moves, than when it is in limiting equilibrium.

76. Motion on a rough plane *A particle slides down a rough plane inclined to the horizon at an angle* α ; *if* μ *be the coefficient of friction, to determine the motion.*

Let m be the mass of the particle, R be the normal reaction of the plane, and μR be the friction. Since there is no motion perpendicular to the inclined plane, the normal reaction must balance the component of mg along the perpendicular to the inclined plane.

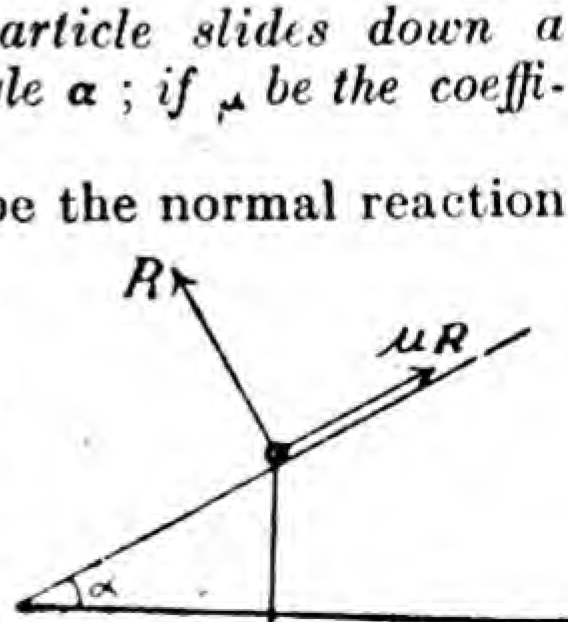


Fig. 46

$$R=mg\cos\alpha \qquad \ldots(1)$$

The total force down the inclined plane is $(mg\sin\alpha-\mu R)$ poundals. If f is the acceleration of the particle, then,

by second Law of motion.

$$mf = mg \sin\alpha - \mu R \qquad \ldots(2)$$

or
$$f = \frac{mg \sin\alpha - \mu R}{m} = g(\sin\alpha - \mu\cos\alpha) \qquad \ldots(3)$$

Hence the velocity of the particle after it has moved from rest over a length l of the plane, is given by

$$v^2 = u^2 + 2fs.$$

In this case $u=0$; $s=l$ and f is given by (3).

$$\therefore \quad v = \sqrt{2gl(\sin\alpha - \mu\cos\alpha)}.$$

Similarly, if the particle were projected up the plane, we have to change the sign of μ, and its acceleration in a direction opposite to that of its motion is $g(\sin\alpha + \mu\cos\alpha)$.

77. *Two equally rough inclined planes, of equal height whose inclinations to the horizon are α and β are placed back to back; two masses, m_1 and m_2, are placed on their inclined faces and are connected by a light inextensible string passing over a smooth pulley at the common vertex of the two planes; if m_1 descend, find the resulting motion.*

Let T be the tension of the string in poundals, R_1 and R_2 the reactions of the planes and μ the coefficient of friction.

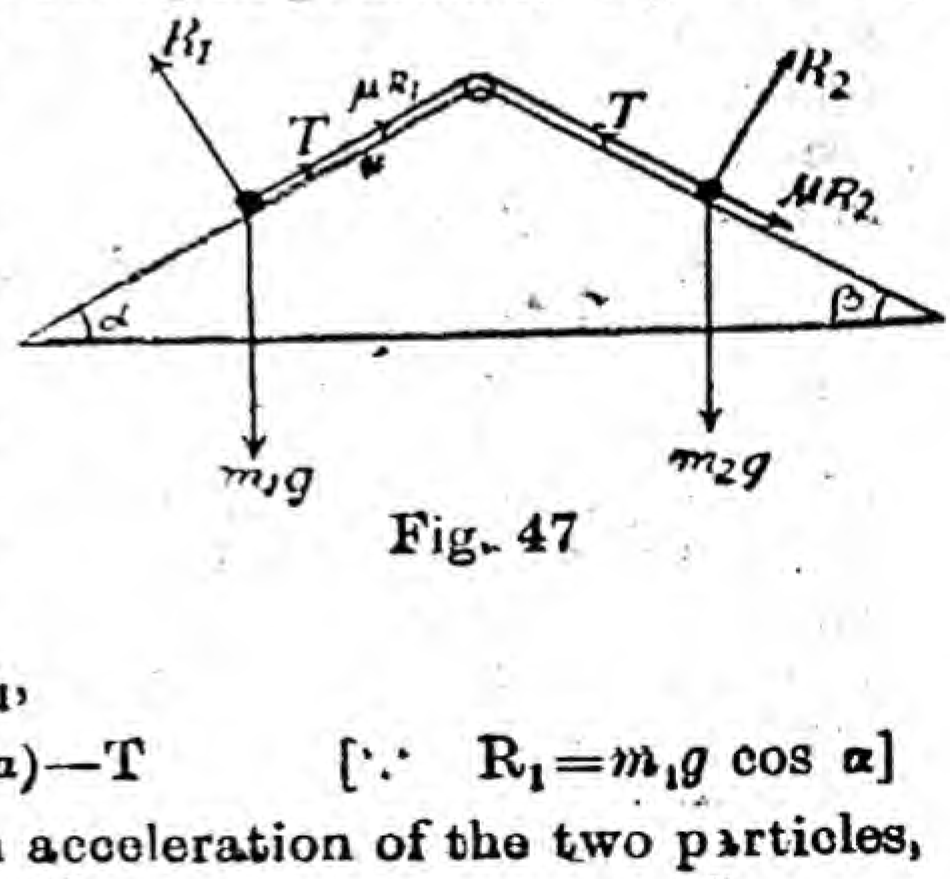


Fig. 47

Since m_1 moves down, the friction on it acts up the plane.

Since m_2 moves up, the friction on it acts down the plane.

Hence the total force on m_1, down the plane

$$= m_1 g \sin\alpha - T - \mu R_1,$$

$$= m_1 g(\sin\alpha - \mu\cos\alpha) - T \qquad [\because R_1 = m_1 g\cos\alpha]$$

Hence, if f be the common acceleration of the two particles, we have, by the 2nd Law of motion,

$$m_1 g(\sin\alpha - \mu\cos\alpha) - T = m_1 f \qquad \ldots(1)$$

Similarly, the total force on m_2 up the plane.

$$=T-\mu R_2-m_2g\sin\beta$$

$$=T-m_2g(\mu\cos\beta+\sin\beta)\qquad[\because\ R_2=m_2g\cos\beta]$$

Hence

$$T-m_2g(\sin\beta+\mu\cos\beta)=m f \qquad ..\ (2)$$

Adding (1) and (2), we have

$$f(m_1+m_2)=g[m_1(\sin\alpha-\mu\cos\alpha)+m_2(\sin\beta+\mu\cos\beta)]$$

giving the required acceleration.

78. *A body of mass m lbs., is placed on a horizontal plane which is in motion with a vertical upward acceleration f; find the reaction between the body and the plane.*

Let R be the reaction between the body and the plane. Since the body is, at any instant, resting on the plane, it is also moving upwards with an acceleration f. Hence, according to Newton's second law, force on the body in the upward direction $=mf$. But R acts upwards, and mg, its weight, acts downwards. Therefore, the force on the body, $R-mg$ acts vertically upwards ; hence

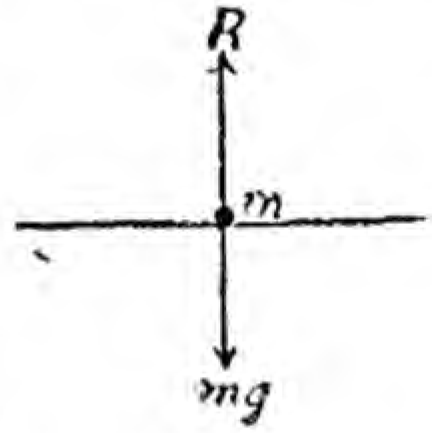


Fig. 48

$$R-mg=mf \text{ or } R=m\,(g+f) \qquad \text{—}\ (1)$$

If, however, the plane be descending with a downward acceleration f, the reaction R_1, is given by

$$mg-R_1=mf$$

or

$$R_1=m\ (g-f). \qquad \ldots\ (2)$$

We note that the reaction is greater or less than the weight of the body according as the acceleration of the body is upwards or downwards. Moreover, R_1 is zero if $f=g$, in the second case.

79. Apparent weight of a man in a lift. When the lift is moving up or down with uniform velocity, $f=0$; hence, the reaction of the floor is equal to the weight of the man. When, however, the lift is being accelerated upwards, the reaction of the floor must be greater than the man's weight, because it has not only to support his weight, but also has to give him an upward acceleration. Again, when the lift is being accelerated downwards, his weight must exceed the reaction of the floor

on his feet by the amount necessary to impart to him the downward acceleration of the lift.

From equation (1) Art. 78, '*the man will seem heavier by the fraction* $\frac{f}{g}$ *of his weight.*' From equation (2) '*the man apparently loses* $\frac{f}{g}$ *of his weight.*'

Ex. 1. A cage weighing $2\frac{1}{2}$ tons is raised and lowered in a coalmine shaft by a steel cable. Find the tension of the cable (1) when the cage is raised or lowered with a constant velocity, (2) when the cage is lowered with the speed increasing uniformly from 0 to 1000 feet per minute, in the first 50 feet.

Let T be the tension in the cable in poundals, just above the cage. The resultant upward force on the cage is,

$$(T-\tfrac{5}{2}\times 2240\times g) \text{ poundals.}$$

(1) If the velocity is constant, the acceleration is zero,

$$\therefore \quad T-\tfrac{5}{2}\times 2240\times g=0$$

$$\text{or} \quad T=\tfrac{5}{2} \text{ tons weight.}$$

(2) If the velocity changes from 0 to 1000 feet per minute, in 50 feet, the acceleration f is given by

$$\left(\frac{1000}{60}\right)^2=2f\times 50$$

$$\text{or} \quad f=\tfrac{25}{9}\text{ ft./sec}^2$$

$$\therefore \quad \tfrac{5}{2}\times 2240\times g-T=\tfrac{5}{2}\times 2240\times \tfrac{25}{9}$$

$$T=\tfrac{5}{2}\times 2240\times g\left(1-\frac{25}{32\times 9}\right)$$

$$=\tfrac{5}{2}\times\tfrac{263}{288} \text{ tons weight}$$

$$=2{\cdot}28 \text{ tons weight (approximately).}$$

Ex. 2. A balloon weighing 800 lbs. is descending with a constant acceleration of 1 ft./sec.^{2}, when 50 lbs. of ballast is suddenly released. Find the magnitude and direction of the acceleration immediately after the release.

Originally the ballast is moving with the plane with a constant acceleration of 1 ft./sec.^{2} The moment it is released,

it begins descending with an acceleration of 32 ft./sec.^{2} Hence the rate of change of momentum of the ballast in the downward direction

$$=50(32-1)$$

$$=50\times 31 \text{ poundals.}$$

The same momentum is communicated to the plane, which now weighs 750 lbs. If f is the acceleration of the plane vertically upwards, we have by Newton's 3rd Law,

$$750\ (f+1)=50\times 31$$

$$+1=\frac{50\times 31}{750}=\frac{31}{15}=2{\cdot}06$$

$$\therefore \qquad f=1{\cdot}06 \text{ ft./sec}^2.$$

Examples VIII

1. A mass of 5 lbs. on a rough horizontal table is connected by a string with a mass of 8 lbs. which hangs over the edge of the table ; if the coefficient of friction be $\frac{1}{2}$, find the resultant acceleration.

Find also the coefficient of friction if the acceleration be half that of a freely falling body.

2. Two rough planes, inclined at 30° and 60° to the horizon and of the same height are placed back to back; masses of 5 and 10 lbs. are placed on the faces and connected by a string passing over the top of the planes ; if the coefficient of friction be $\frac{1}{\sqrt{3}}$, find the resulting acceleration.

3. A rough plane is 100 feet long and is inclined to the horizon at an angle $\sin^{-1}\frac{3}{5}$, the coefficient of friction being $\frac{1}{3}$, and a body slides down it from rest at the highest point ; find its velocity on reaching the bottom.

If the body were projected up the plane from the bottom so as just to reach the top, find its initial velocity.

4. The velocity of flow in a water-main of 6 inches diameter is 5 ft./sec. At one place the main is bent through an angle of 30°. Find the resultant force on the bend.

5. Water issuing from a nozzle of 2 inches diameter, with a velocity of 50 feet per second impinges on a vertical wall, the jet being at right angles to the wall. If there is no splash find the pressure exerted on the wall.

6. Fifty cubic feet of water are flowing per minute along the fixed vane AB. The speed along the vane is constant and equal to 20 ft. per second. Find the magnitude and direction of the resultant force produced on the vane.

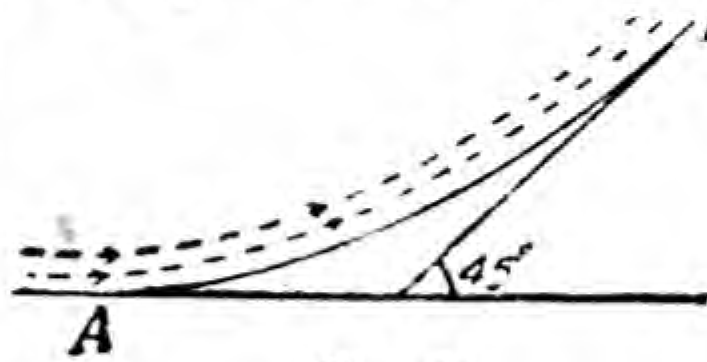


Fig. 49

7. A loaded cage weighing 2 tons is drawn up a mine shaft by a wire rope passing over a pulley at the top. If the pull is constant, and the cage acquires a speed of 10 feet per second after it has been rising for 5 seconds, find the tension in the rope. If men weigh 240 stone, find the pressure they cause on the floor of the cage in tons.

8. A balloon is rising with an acceleration f. Prove that the fraction of the weight of the balloon which must be emptied out in the form of sand in order to double this acceleration is $\frac{f}{2f+g}$, assuming the upthrust of air to remain unaltered and that air resistance is neglected.

9. A balloon of total mass 620 lbs. is drifting horizontally when 40 lbs. of sand are suddenly released. Find the acceleration immediately after the sand is released.

10. A balloon ascends vertically with a uniformly accelerated motion so that a weight of 1 lb. produces on the hand of the aeronaut sustaining t, a downward pressure equal to that which 17 oz. produces when at rest; find the height which the balloon attains in one minute from rest.

11. A man in a lift at rest holds in his hand 1 lb.; suddenly 1 lb. appears to weigh 15 oz; then suddenly the weight appears to change to that of 17 oz.; and next it appears to have its usual weight; he finds that the lift has descended 128 ft. from rest and has come to rest. How long did it take in the descent?

12. A train of mass 140 tons, travelling at the rate of 15 miles per hour, comes to the top of an incline of 1 in 128, the length of the incline being half a mile and then the steam is shut off; taking the resistance due to friction, etc., as 10 lbs. wt. per ton, find the distance it describes on a horizontal line at the foot of the incline before coming to rest.

13. A locomotive exerts a constant drawbar pull of 3 tons on a train weighing 150 tons, moving up a 1 per cent grade. Resistance other than gravity amount to 12 lb. per ton. How long will it take to change the speed from 15 miles per hour to 30 miles per hour, and how far will the train move in that time?

14. A mass m is drawn up a smooth inclined plane, of height h and length l, by means of a rope passing over the vertex of the plane, from the other end of which hangs a mass m'. Show that, in order that m may just reach the top of the plane, m' must be detached after m has moved a distance

$$\frac{m+m'}{m'} \cdot \frac{hl}{h+l}.$$

CHAPTER VII

IMPULSE, WORK, POWER AND ENERGY

80. Impulse of a Force. If a force P of constant magnitude acts on a particle for a time t, the product $P \times t$ is called the impulse of the force for the time t. If P is measured in pounds-weight, the unit of impulse is "pounds weight-seconds". If P is measured in poundals, the unit becomes "poundal seconds".

If, however, the force is variable, its impulse in a given time is equal to the product of the mean value of the force and the given time.

81. Relation between Impulse and Momentum. Suppose a force P acts on a mass m for t seconds and changes its velocity from u to v, and generates in it an acceleration f, then

$$
\begin{aligned}
I &= P \times t \\
&= mf \times t \\
&= m(v-u) \\
&= mv - mu \\
&= \text{change in momentum.}
\end{aligned}
$$

It is to be noted that P should be in poundals, because, otherwise, we cannot write $P = mf$.

82. Blows or Shocks. We often come across forces which are enormously great in magnitude but act for a very short time. In such cases the effect of the force is measured by the change of momentum produced, which may be finite. The blow of the hammer, or the forces arising when the cricket ball is hit hard by the bat, are the forces which come in this category, and are also known as impulsive forces.

Ex. A hammer whose mass is 2 bs. strikes a fixed steel plane with a velocity of 10 ft./sec; if the hammer rebounds after $\frac{1}{10000}$ second with a velocity of 6 ft./sec. what is the mean value of the pressure exerted on the plane ?

$$\begin{aligned} \text{Impulse} &= \text{change in momentum} \\ &= 2\times 10 - 2\times(-6) \\ &= 32 \text{ units of momentum.} \end{aligned}$$

But $I = P\,t$

or $P\,\frac{1}{10000} = 32$

or $P = 320000$ poundals

$\therefore$ Pressure $= \frac{320000}{32} = 10000$ lbs. wt.

83. Motion of a Shot and Gun. When a gun is fired, the powder is almost immediately ignited and converted into a gas at a very high pressure. The gas expands and forces the shot out of the gun with a great velocity. The action of the gas is similar to that of a compressed spring trying to regain its original shape. The force exerted on the shot forward is, at any instant before the shot leaves the gun, equal and opposite to that exerted on the gun backward. Hence the impulse of the force on the shot is equal and opposite to the impulse of the force on the gun. Hence the momentum generated in the shot is equal and opposite to that generated in the gun, if the latter is free to move. This is precisely the reason for keeping the butt tight close to the shoulder near the arm-pit while firing a gun or a rifle. By holding the gun in this manner the backward motion of the gun is shared by the whole body and less jerk is felt.

Ex. 1. A shot, whose mass is 200 lbs., is projected from a gun of mass 40 tons, with a velocity of 800 feet per second; find the velocity generated in the gun.

If v is the velocity generated in the gun, and since the momentum of the gun is equal and opposite to that of the shot, we have

$$40\times 2240\times v = 200\times 800$$

$$\therefore \qquad v = 1\tfrac{11}{14} \text{ ft./sec.}$$

Ex. 2. A shot, whose mass is 800 lbs, is discharged from an 81-ton gun with a velocity of 1400 ft./sec.; find the constant force which acting on the gun would stop it after a recoil of 5 feet.

If v is the velocity of recoil, we have

$$81\times 2240\times v = 800\times 1400.$$

If f is the retardation due to which the gun stops after a recoil of 5 feet,

$$0=\left[\frac{800\times 1400}{81\times 2240}\right]^2-2f\times 5.$$

Hence the required force

$$P=\text{Mass of the gun}\times\text{retardation}$$

$$=81\times 2240.\,\tfrac{1}{10}\left[\frac{800\times 1400}{81\times 2240}\right]^2 \text{ poundals}$$

$$=9\tfrac{209}{324}\text{ tons' wt.}$$

Ex. 3. A gun is mounted on a gun carriage movable on a smooth horizontal plane, and the gun is elevated at an angle α to the horizon; a shot is fired and leaves the gun in a direction inclined at an angle θ to the horizon; if the mass of the gun and its carriage be n times that of the shot, show that

$$\tan\theta=\left(1+\frac{1}{n}\right)\tan\alpha.$$

Let u be the velocity of the shot along the barrel, relative to the gun, and v the backward horizontal velocity of the gun and the carriage. Let m be the mass of the shot, and nm that of the gun and carriage. The resultant velocity of the shot at an angle θ to the horizon is compounded of u at an elevation α, and v horizontally backwards.

Equating the impulse on the gun in the horizontal direction to that of the shot in the same direction, we get

$$nmv=m(u\cos\alpha-v) \qquad \ldots\ldots (1)$$

where $u\cos\alpha-v$ is the actual velocity of the shot in the horizontal direction

If V is the resultant velocity of the shot inclined at an angle θ to the horizon,

$$V\cos\theta=u\cos\alpha-v \qquad \ldots\ldots (2)$$

and

$$V\sin\theta=u\sin\alpha \qquad \ldots\ldots (3)$$

Dividing (3) by (2), and using (1)

$$\tan\theta=\frac{u\sin\alpha}{u\cos\alpha-v}=\left(1+\frac{1}{n}\right)\tan\alpha.$$

Ex. 4. A hammer head weighing 1·2 lbs., and moving with a velocity of 16 feet/sec., strikes a nail of 0·1 lb. weight

and drives it $\frac{1}{2}$ inch into a piece of wood. Assuming no rebound of the hammer and the resistance to penetration of the nail constant find its magnitude.

Let v be the velocity with which the nail and hammer move immediately after the impact. Then by the law of conservation of momentum

$$1{\cdot}2 \times 16 = (1{\cdot}2 \div 0{\cdot}1)\, v$$

$$\therefore \quad v = \frac{16 \times 1{\cdot}2}{1{\cdot}3}$$

$$= 14{\cdot}8 \text{ feet per second.}$$

Since the resistance to penetration is constant the time rate of change of velocity will be uniform.

$$\therefore \quad \text{average velocity of penetration} = \frac{14{\cdot}8}{2}$$

$$= 7{\cdot}4 \text{ ft./sec.}$$

" → inches
' → foot

$$\text{and the time of penetration} = \frac{1}{24 \times 7{\cdot}4} \text{ second.}$$

$1'' = \frac{1}{12}'$
$\frac{1}{2}'' \to \frac{1}{12} \cdot \frac{1}{2} = \left(\frac{1}{24}\right)$

The resistance = rate of change of momentum

$$= 1{\cdot}3 \times 14{\cdot}8 \times 24 \times 7{\cdot}4 \text{ poundals}$$

$$= 107 \text{ lbs. wt.}$$

The magnitude of the blow between the hammer and the nail = the change of momentum of the hammer

$$= 1{\cdot}2(16 - 14{\cdot}8)$$

$$= 1{\cdot}2 \times 1{\cdot}2$$

$$= 1{\cdot}44 \text{ F.P.S. units.}$$

Ex. 5. A machine-gun is mounted on an aeroplane and when the latter is travelling at 50 m. p. h. the gun is fired in the direction of travel for 15 seconds. Find the reduction in speed of the aeroplane due to this, and the force tending to move the gun relative to the aeroplane.

Total weight of the aeroplane 1800 lbs. Rate of firing 600 bullets per minute. Weight of bullet $\frac{1}{2}$ oz. Muzzle velocity of bullets 2000 ft./sec.

The reaction of the force acting on the bullets is at every instant acting on the gun, and therefore on the aeroplane.

The time average of the force on the bullets = the change in momentum per second.

The mass discharged per second $= \dfrac{600}{2 \times 16 \times 60} = \dfrac{5}{16}$ lb.

The change of velocity $= 2000$ ft./sec.

$\therefore$ Force $= \frac{5}{16} \times 2000$ poundals

$$= \frac{5 \times 2000}{16 \times 32} = 19{\cdot}5 \text{ lbs. wt.}$$

This is the force tending to move the gun relative to the aeroplane. Now let us examine the effect of the reaction of the force on the aeroplane. We may neglect the small change of mass due to the discharge of the bullets.

The force of $\frac{5}{16} \times 2000$ poundals acts for 15 seconds. If v be the velocity of the aeroplane in feet per second, at the end of 15 seconds, we have

$$1800 \left(\frac{50 \times 88}{60} - v \right) = \frac{5}{16} \times 2000 \times 15$$

or $$v = 68{\cdot}1 \text{ feet./sec.}$$

Examples IX

1. A train of 100 tons is observed to be moving on smooth rails with a velocity of 15 miles an hour ; after an interval it is observed to be moving at 45 m.p.h. ; what impulse has been applied to it ?

2. A football of $\frac{1}{2}$ lb. wt. is moving just after it has been kicked from rest with 20 ft./sec ; what impulse it has received and find the force of the impulse, supposing the interval of the impulse to have been (*i*) $\frac{1}{2}$ sec. (*ii*) $\frac{1}{10}$ sec.

3. A tricyclist and his machine are 2 cwt. ; find the pressure on the pedals (neglecting friction, the rotary motion of the wheels etc.), supposing that he attains a velocity of 15 m. p. h. from rest in 20 seconds, and that he applies pressure for two-thirds of each revolution of the wheels.

4. A sculler can apply to the water with the blades of his sculls a pressure of 28 lbs. wt. ; supposing he rows 30 strokes a minute and that the pressure is applied to the water for a quarter of each stroke, find what impulse he applies in one minute.

5. A hammer of 2 lbs. hits a nail with a horizontal velocity of 40 ft./sec. and drives the nail into a board to the depth of half an inch ; find (*i*) the force which is exerted on the nail supposing it uniform, (*ii*) the duration of the impulse.

6. A shot, of mass 700 lbs., is fired with a velocity of 1700 feet per second from a gun of mass 38 tons ; if the recoil be resisted by a constant force equal to the weight of 17 tons, through how many feet will the gun recoil ?

84. Work. If a force acting on a body causes a displacement of it then the force is said to do work. The work done is positive when the resolved part of the displacement vector in the direction of force has the same sense as the force. The work done is negative if the resolved part of the displacement vector has a sense opposite to that of the force.

When a particle falls towards the earth, the work done is positive, because the particle moves in the same direction in which the force is acting. But, in the case of a particle thrown up from the surface of the earth, the work done during its ascent is negative.

We see that there are two things necessary before work can be done, *viz.* force and motion. Due to its dependence on these two quantities, the word work acquires a technical significance, and should be more properly called "mechanical work."

85. Measure of work. *The quantity of work done is measured by the product of the force and the resolved part of the displacement vector in the direction of force.*

If the point of application moves from A to B,

Work done $= P \times s . \cos \theta$,

where $AB = s$.

$= P \cos \theta \times s$.

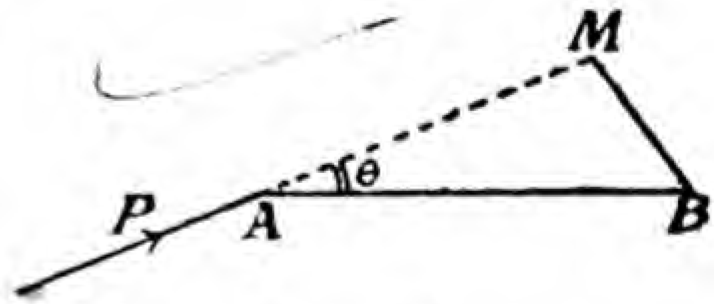


Fig. 50

The work done is also equal to the component of the force in the direction of the displacement multiplied by the displacement.

86. Unit of work. The absolute unit of work in the F.P.S. system is *foot poundal.*

One foot-poundal is the work done by 1 poundal of force acting through a distance of 1 foot. In C. G. S. system the absolute unit is the erg. One erg is the work done by 1 dyne of force acting through a distance of 1 centimetre.

1 Joule $= 10^7$ ergs.

87. Gravitation units of work. The unit of work used by engineers is a Foot-pound, which is the work done in raising the weight of one pound through one foot. Since the weight of a pound is equal to g poundals, it follows that a Foot-Pound is equal to g-Foot-Poundals.

Similarly in the C. G. S. system, the unit of work is gramme-centimetre, *i.e.* the work done in raising a mass of one gramme through a vertical height of one centimetre.

88. Power. *The power of an agent is the rate of its doing work, and is measured by the amount of work done by the agent working uniformly for a unit of time.*

If F is the force in lbs. wt. and s the resolved part of the displacement vector along the line of action of the force, the amount of work done is $P \times s$ units of work. If t is the time taken in seconds to accomplish work,

$$\text{The rate of work} = F \times \frac{s}{t}$$
$$= F \times v \text{ foot lbs./sec.}$$

where v is the velocity of the point of application in the direction of the force.

89. Unit of Power. The unit of power in F. P. S. system is a foot-pound per second, while in the C. G. S. system it is one Joule per second, and is called a Watt.

1 Watt $=$ 1 Joule per second $= 10^7$ ergs per second.

90. Horse power. *A Horse-power (H.P.) is the unit of power (commonly used by engineers) equal to 33000 ft. lbs. of work per minute or 550 ft. lbs. of work per second.*

If a body under the action of a force F lbs. wt., move with velocity v ft./sec., then the Horse-power acting on it is given by

$$H = \frac{F \times v}{550}.$$

Ex. 1. A train of 100 tons is pulled by an engine on the level at a constant speed of 30 m.p.h, the resistance due to friction etc. being 10 lbs. per ton. Find the minimum horse-power of the engine. Find also the acceleration of the train when it is moving at 10 m.p.h., the engine working at its minimum rate.

The resistance $= 100 \times 10 = 1000$ lbs.

Speed $= 30$ m.p.h $= 44$ ft./sec.

Since the train is moving at a constant rate, the pull of the engine is equal to the resistance.

$\therefore$ The rate at which the engine is working

$= 1000 \times 44$ ft. lbs. per sec.

$\therefore$ Horse power of the engine $= \dfrac{44000}{550} = 80.$

In the second part of the question, the train starts from rest and the engine is all the time working at 80 H. P. It will accelerate the motion till the velocity becomes 30 m.p.h.

Here the velocity is 10 m.p.h. $= \dfrac{44}{3}$ ft./sec.

If F is the force exerted by the engine, we have

$$F \times \frac{44}{3} = 80 \times 550$$

$\therefore$ $F = 3000$ lbs.

Resistance $= 1000$ lbs.

If f is the acceleration at that instant

$$mf = (F - R)\ 32$$

$$\therefore \quad f = \frac{(3000 - 1000) \times 32}{100 \times 2240}$$

$$= \tfrac{2}{7} \text{ ft./sec}^2.$$

Ex. 2. A lift weighing 5 cwt. rises from rest through a height of 50 feet in 5 seconds with uniform acceleration. Find the average horse-power exerted during this time.

If f is the acceleration of the lift, then

$$s = \tfrac{1}{2}\,(f - g)\,t^2$$

$\therefore$ $50 = \tfrac{1}{2}\,(f - g)\ .25$

$\therefore$ $f - g = 4$

or $f = g + 4 = 36$ ft./sec².

Now, the average velocity $= \dfrac{\text{total distance}}{\text{total time}} = \dfrac{50}{5}$

$= 10$ ft./sec.

Force on the lift $= 5 \times 112 \times 36$ poundals

$$= \frac{5 \times 112 \times 36}{32} \text{ lbs. wt.}$$

$$\therefore \quad \text{Average H. P.} = \frac{\text{Force} \times \text{Average velocity}}{550}$$

$$= \frac{5 \times 112 \times 36 \times 10}{32 \times 550}$$

$$= 11\tfrac{5}{11}.$$

Examples X

1. A horse draws a tram-car of 2 tons along a horizontal road with the uniform velocity of $7\frac{1}{2}$ m p.h ; supposing the resistance due to friction etc., is $\frac{1}{[illegible]0}$ of the weight of the car, find how many foot lb. the is doing per second.

2. Find the H.P. of an engine which can travel at the rate of 25 mills per hour up an incline of 1 in 100, the mass of the engine and load being 10 tons, and the frictional resistance being 10 lbs wt per ton.

3. A locomotive, exerting 500 H.P., is hauling a train whose total weight is 100 tons, up a slope of 1 in 100, the friction being 20 lbs. wt. per ton. If the speed be 30 miles per hour, find the acceleration.

4. An ocean steamer does n knots when the engines indicate N. H.P.·Find, in tons, the resistance of the steamer in her passage through water. (1 knot=6086 feet per hour).

5. What is the horse-power of an engine which can project 10,000 lbs, of water per minute with a velocity of 80 feet per second, 20 per cent. of the whole work done being wasted by friction ?

6. A railway siding is level for the first 50 yds., and then rises at a slope of $\frac{1}{40}$. A wagon weighing 10 tons is shunted on to the siding with a velocity of 18 miles per hour, and is observed to reach the foot of the incline in 6 seconds. If the resisting force due to friction is constant, what is its magnitude ?

How far up the incline will the wagon travel before coming to rest ?

91. Energy. The energy of a body is its capacity for doing work and is of two kinds, Kinetic and Potential.

92. The Kinetic Energy *of a body is the energy which it possesses by virtue of its motion, and is measured by the amount of work that the body can perform against the impressed forces before its velocity is destroyed.*

If a particle is moving with a constant velocity v, and no force acts on it, it will continue moving uniformly in a straight line. But if a constant force P poundals acts in a direction opposite to that of its motion, the motion of the particle is gradually slowed down and after some time the particle comes

to rest. The force produces an acceleration $-f$, given by $P=mf$, where m is the mass of the moving particle.

If x is distance moved by the particle before coming to rest, we have

$$0=v^2-2fx$$

or $$fx=\tfrac{1}{2}v^2.$$

$\therefore$ $$Px=mfx=\tfrac{1}{2}mv^2 \text{ ft. poundals.}$$

Hence the kinetic energy of the particle = work done by it before it comes to rest.

93. Relation between the Kinetic Energy and the work done. In a uniformly accelerated motion, if the initial and the final velocities are u and v and s is the space described by the body, we have

$$v^2=u^2+2fs.$$

But $$P=mf.$$

Hence the work done by the particle in changing its speed from u to v

$$\begin{aligned} &=Ps=mfs \\ &=m\tfrac{1}{2}(v^2-u^2) \\ &=\tfrac{1}{2}mv^2-\tfrac{1}{2}mu^2 \\ &=\text{change in K.E.} \end{aligned}$$

Therefore $$P=\frac{\tfrac{1}{2}mv^2-\tfrac{1}{2}mu^2}{s}$$

$$=\text{change in K.E. per unit of space.}$$

Hence force may also be defined as the space rate of change of Kinetic Energy.

94. The Potential Energy *of a body is the work it can do by means of its position in passing from its present configuration to some standard configuration (usually called its zero positon).*

A body raised to a height above the surface of the earth has potential energy. Water stored up in a reservoir, when let out through a fountain rises up in the air. A piece of stone as it falls from a height to the earth's surface can do work. The surface of the earth is usually taken as a standard configruation, when the P.E. of a body is zero.

95. *To show that the sum of the kinetic and potential energies of a particle of mass m is constant throughout the motion when it falls from rest at a height h above the ground.*

$AH = h$

Let v be the velocity when it has fallen a distance $HP = x$

Its K.E. at $P = \frac{1}{2}mv^2$

$= \frac{1}{2}m.2gx = mgx$

Its P.E. at P = The work its weight can do as it falls through the distance PA

$= mg.PA = mg(h - x)$

$\therefore$ K.E. + P.E. $= mgx + mg(h-x)$

$= mgh.$

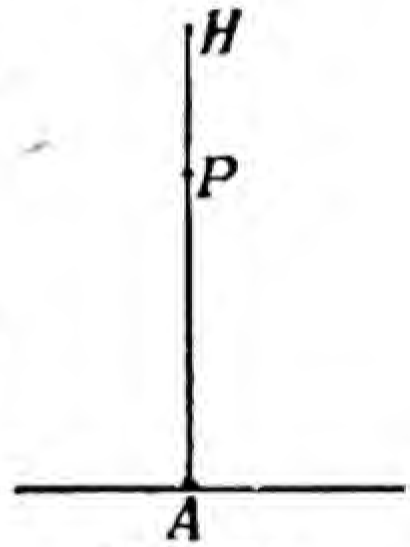


Fig. 51

When the particle is at H, its K.E. is zero there, and its P.E. is mgh. As it falls down its kinetic energy increases and the potential energy decreases, so much so that at the point A, the K.E. becomes mgh, and P.E. becomes zero.

96. Law of Conservation of Mechanical Energy. When a particle of mass m ascends a height h, the work done by gravity is $-mgh$ and when it comes back to the surface of the earth the work done is $+mgh$. Thus the total work done in the two cases is zero. Such is not the case with all forces. If a body is dragged through a distance s against a constant frictional force F, the work done is Fs. To bring the body back to its former position on the same path an equal amount of work Fs is to be done. Thus the total work is $2Fs$ and not zero. *Only when the total work done by a force during its displacements till it returns to its original position, is zero, the sum of the kinetic and potential energies remains constant.* In other cases energy is always converted to various forms *e.g.*, heat, light and sound.

In its most general form, the principle of conservation of energy says :

The total amount of energy in the universe is constant ; energy cannot be created or destroyed although it may be converted into various forms.

Ex. 1. A bullet of mass 2 ounces, is fired into a target with a velocity of 1280 feet per second. The mass of the

target is 10 lbs. and it is free to move ; find the loss of K.E. by the impact in foot pounds.

$$\text{K.E. of the bullet} = \frac{1}{2} \cdot \frac{1}{8} (1280)^2 \text{ units of energy.}$$

Using the law of conservation of momentum to calculate the velocity of the target and the shot after hit, we get

$$\frac{1}{8} . 1280 = \left(10 + \frac{1}{8} \right) v,$$ where v is the velocity of the target and the shot.

$$v = \frac{1280}{81} \text{ ft./sec.}$$

$$\text{K.E. of the shot and the target} = \frac{1}{2} \cdot \frac{81}{2} \left(\frac{1280}{81} \right)^2 \text{ units of K.E.}$$

$$\therefore \quad \text{Loss of K.E.} = \frac{1}{16} (1280)^2 \left[1 - \frac{1}{81} \right] \text{ units of K.E.}$$

$$= \frac{1280 \times 1280 \times 80}{16 \times 81 \times 32} \text{ foot lbs.}$$

$$= 3160 \frac{40}{81} \text{ ft. lbs.}$$

Ex. 2. A hammer, of mass M lbs. falls from a height h feet upon the top of a pile, of mass m lbs. and drives it into the ground a distance a feet ; find the resistance of the ground, it being assumed to be constant, and the pile being supposed inelastic.

Find also the time during which the pile is in motion and the K.E. lost at the impact.

Let u be the velocity of the hammer on hitting the pile, so that $u^2 = 2gh$ ...(1)

Let v be the velocity of the hammer and pile immediately after the impact. By the principle of conservation of momentum,

$$(M + m)v = Mu \qquad ...(2)$$

If P be the resistance of the ground in poundals, the force to resist the driving of the pile into the ground

$$=P-(M+m)g$$

The Principle of Conservation of Energy gives

$$\tfrac{1}{2}(M+m)v^2=[P-(M+m)g].\,a$$

$$\therefore \qquad P=(M+m)g+(M+m)\frac{v^2}{2a}$$

$$=(M+m)g+\frac{M^2}{M+m}\,\frac{u^2}{2a}, \quad \text{using (2)}$$

$$=(M+m)g+\frac{M^2}{M+m}g\,\frac{h}{a}. \quad \text{using (1)}$$

A weight of slightly more than $\frac{M^2}{M+m}\cdot\frac{h}{a}$ lbs. placed on the pile would thus slowly overcome the resistance and just drive the pile down.

The Principle of Momentum gives the time t during which the pile is in motion. For

$$[P-(M+m)g]\times t=\text{change in the momentum}$$

$$=(M+m)v=Mu,$$

$$\therefore \qquad t\times\frac{M^2}{M+m}\,\frac{u^2}{2a}=Mu$$

or

$$t=\frac{M+m}{M}\cdot\frac{2a}{u}=\frac{M+m}{M}\,a\sqrt{\frac{2}{gh}}.$$

The K.E. lost at the impact

$$=\tfrac{1}{2}Mu^2-\tfrac{1}{2}(M+m)v^2$$

$$=\tfrac{1}{2}Mu^2-\tfrac{1}{2}\,\frac{M^2}{M+m}\,u^2$$

$$=\tfrac{1}{2}\,\frac{Mm}{M+m}\,u^2$$

$$=\frac{m}{M+m}.\,\tfrac{1}{2}Mu^2$$

$=\frac{m}{M+m}\times$ energy of the hammer on striking the pile.

Ex. 3. A belt of leather is moving horizontally at 12 ft./sec. A slab of iron weighing 10 lbs. is gently laid on the belt, the coefficient of friction between the belt and iron being 0·3. Calculate

(*a*) the time taken for the slab to acquire the velocity of the belt ;

(*b*) the distance through which the belt has been moved relative to the slab ;

(*c*) the total amount of extra energy supplied by the prime mover driving the belt. What happens to this energy ?

The frictional force which accelerates the slab

$$=0{\cdot}3\times10=3 \text{ lb. wt.}$$

∴ The acceleration of the slab

$$3\times32=10\times f$$

or

$$f=\tfrac{48}{5} \text{ ft./sec}^2.$$

Time taken by the slab, starting with zero velocity to acquire a velocity of 12 ft./sec., subject to the above acceleration. is given by $v=u+ft$

$$12=0+\tfrac{48}{5}t$$

or

$$t=\frac{12\times5}{48}=\tfrac{5}{4} \text{ sec.}.$$

During this time the distance moved by the belt

$$=\frac{12\times5}{4}=15 \text{ ft.}$$

The distance moved by the slab is

$$\begin{aligned} s&=ut+\tfrac{1}{2}ft^2\\ &=0+\tfrac{1}{2}.\tfrac{48}{5}.\tfrac{25}{16}\\ &=7\tfrac{1}{2} \text{ ft.}\end{aligned}$$

∴ The distance moved by the belt relative to the slab

$$=15-7\tfrac{1}{2}=7\tfrac{1}{2} \text{ ft.}$$

Extra energy supplied

$$\begin{aligned} &= mv^2\\ &=\tfrac{1}{2}.10\times144 \text{ energy units}\\ &=\frac{5\times144}{32}=22\tfrac{1}{2} \text{ ft. lb.}\end{aligned}$$

Examples XI

1. Find the average force which will bring to rest, in 2 feet, an ounce bullet, moving at the rate of 1500 ft./sec. How long will it take to bring it to rest ?

2. A stone, moving with a velocity of 15 ft./sec., would just break through a pane of glass and come to rest. If the same stone be allowed to strike the pane with a velocity of 17 ft./sec. what will be its velocity after passing through ?

3. A shot of mass 28 lbs. is fired from a gun of mass one ton, which recoils up a smooth inclined plane, rising to a height of 5 feet ; find the initial velocity of the projectile.

4. An engine pumps water from a well to the ground level which is 44 feet above the mean level of the water surface. If 2 cubic feet of water is raised per second, and two-thirds of the work of the engine is used in lifting the water, what is the horse-power developed by the engine ?

5. A pile driver of mass 2 cwt. falls from a vertical distance of 16 feet and strikes a pile of mass 10 cwt. which it drives a distance of 3 inches into the ground. Assuming that after the blow the pile driver and the pile meve together, calculate the resistance, supposed to be uniform, which the ground offers in tons weight.

6. A block of mass 16 lb. is moving on a smooth horizontal table with a velocity of 10 feet per second. A force of 3 lb. wt. is applied in a direction opposite to its motion and acts until the block has reversed its direct on and is moving with a velocity of 2 ft./sec. Find the time for which the force acts and the distance of the block from its starting point after this time.

7. A ball wei hing 4 oz. and moving at the rate of 20 feet per second is struck by a bat and rebounds with a velocity of 44 feet per second. Find in foot-lbs. the work done on the ball, and in lbs-wt. the average pressure on the bat, assuming the bat and the ball to be in contact for 0·1 sec.

8. A shell of mass M is moving with a velocity u in the line AB. An internal explosion which generates an energy E breaks it into two masses m_1 and m_2 which move in the line AB. Show that their velocities are $u+\sqrt{\frac{2m_2E}{m_1M}}$ and $u-\sqrt{\frac{2m_1E}{m_2M}}$.

9. A shell moving at 60 ft./sec. bursts into parts of masses 36 lbs. and 6 lbs. If the larger piece continues moving in the original direction at 75 ft./sec., what is the velocity and direction of the other piece ? Calculate the K.E. of the shell before and after explosion.

10. An inelastic pile of weight w is driven into the ground by a striker of weight W which falls on it. When at each stroke the striker falls a clear distance of 11 ft. and $W=5w$, it is found that 11 strokes are necessary to drive the pile 10 inches. How many strokes are necessary if the weight of the striker is doubled ?

11. A bullet of mass $\frac{1}{2}$ lb. is fired horizontally from a gun of mass one ton, the gun carriage resting on the level ground. If the muzzle velocity of the bullet is 560 ft./sec calculate the momentum and K.E. (*a*) of the bullet, (*b*) of the gun, at the instant the bullet leaves the muzzle. If the carriage recoils 6 inches, calculate the mean frictional force exerted by the ground.

CHAPTER VIII

PROJECTILES

97. In the previous chapters we considered motion in straight lines, where the acceleration acted along the line of motion of the particle. We now pass on to the case when the acceleration no longer acts in the direction of motion of the particle. The path which the particle describes is a certain curve, and is no longer a straight line. A particle is projected into the air in any direction inclined to the vertical and with any velocity. We neglect the resistance of air and consider the motion to be in vacuo, so that, there is no other force except the weight of the projectile acting on it during the motion. Moreover, the projectile is always within such a distance of the earth's surface, that the acceleration due to gravity may be considered to remain sensibly constant. This treatment is far divorced from reality and is of little practical value, but it is considerably simple.

98. Definitions. The angle that the direction of projection of a projectile makes with the horizontal plane through the point of projection is called the *angle of projection.*

The path described by a projectile is called its *trajectory.*

The distance between the point of projection and the point at which the projectile strikes a given plane through the point of projection is called its *range* on that plane.

The time taken by the projectile to return to the horizontal plane through the point of projection is called the *time of flight.*

99. According to the first law of motion, if a particle be projected in vacuo or into the air, it should move in a straight line. But experience shows that the particle describes a curved path. Newton explained this departure from a straight path by bringing into the field his hypothesis of gravitation ; that the earth exerts on the particle an attractive force, which interfering with the motion of the particle in a straight line causes it to describe the curved path.

The hypothesis of gravitation, though quite simple in its nature and application, raises certain questions of a more fundamental nature. It was questioned by the German philosopher and physicist Ernst Mach and has been challenged by Albert Einstein even in our days.

100. *A body is thrown with a given velocity u in any given direction. To construct geometrically its position at any given instant of the motion.*

Let AP be the direction of projection. On AP cut off $AB = ut$ = distance which would be traversed in time t, if the velocity were uniform and equal to u.

Draw AD vertically downwards and make $AD = \frac{1}{2}gt^2$ = distance which would be traversed in time t by a bodyfalling freely from rest under gravity. Complete the parallelogram ABCD. Then C represents the actual position of the body at time t. This construction may be used to find the position of the body at every second of its motion. The curve obtained on joining these points gives the path of the body. In the above treatment we have used the law of Physical Independence of Forces (Art. 65).

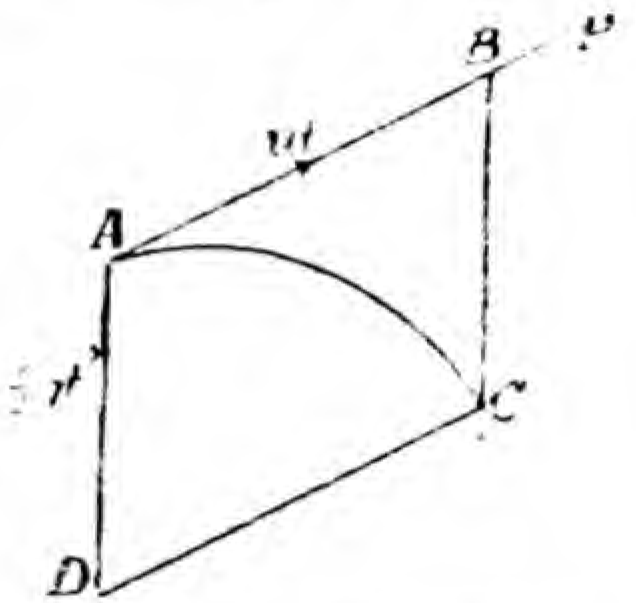


Fig. 52

To prove that, let us suppose that an imaginary particle of equal mass, *but without weight*, starts from A at the same instant and with the same velocity. Then, by Newton's First law, this imaginary particle will travel along the line AP with uniform velocity u and after a time t will reach the point B, such that $AB = ut$.

But the difference between the motion of this imaginary particle and the projectile is due solely to the fact that the latter is being acted upon by a force mg acting vertically downwards. Hence, relative to the imaginary particle the projectile is falling freely under gravity.

Hence after a time t the projectile will be at a distance $\frac{1}{2}gt^2$ below the imaginary particle. But the imaginary particle is at B, and $BC = AD = \frac{1}{2}gt^2$. Hence by the law of parallelogram of velocities the projectile is at C.

101. Let a heavy particle be projected from P, with an initial velocity u making an angle α with the horizontal plane. If we suppose the motion to be in vacuo, the only force acting on the particle is its weight acting vertically downwards. By the Principle of Physical Independence of Forces (Art. 65) the weight of the body produces its effect solely in the vertical direction, and has no effect in the horizontal direction. The horizontal component of velocity of the particle, therefore, remains constant throughout its motion, and is equal to $u \cos \alpha$.

The vertical component of the initial velocity is $u \sin \alpha$, and the acceleration in the vertical direction upwards is $-g$.

102. *To find the velocity and direction of motion after a given time has elapsed.*

Let v be the velocity, and θ the angle which the direction of motion at the end of time t makes with the horizontal.

Then, $v \cos \theta =$ horizontal velocity at the end of time t
$= u \cos \alpha$.

Also, $v \sin \theta =$ the vertical velocity at the end of time t
$= u \sin \alpha - gt$.

Hence, by squaring and adding

$$v^2 = u^2 - 2ugt \sin \alpha + g^2t^2$$

and, by division

$$\tan \theta = \frac{u \sin \alpha - gt}{u \cos \alpha}.$$

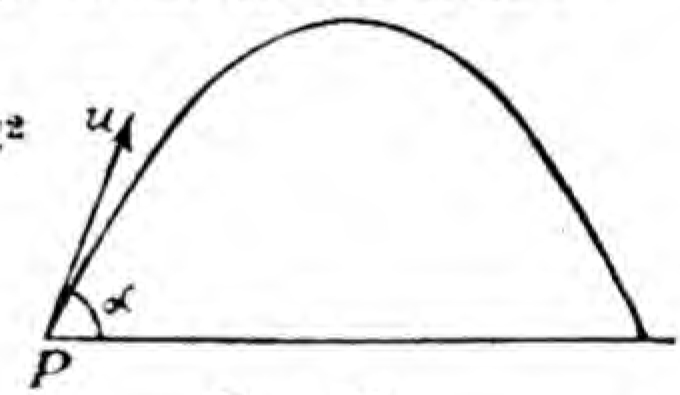


Fig. 53

103. *To find the velocity and direction of motion at a given height h.*

Let v be the magnitude, and θ the inclination to the horizon, of the velocity of the particle at a given height h.

Then, $\quad v \cos \theta = u \cos \alpha$

and, $\quad v \sin \theta = \sqrt{u^2 \sin^2 \alpha - 2gh}$

Squaring and adding,

$$v^2 = u^2 - 2gh$$

and, by division

$$\tan \theta = \frac{\sqrt{u^2 \sin^2 \alpha - 2gh}}{u \cos \alpha}.$$

Hence the velocity of the particle at a given height h is the same as that of a similar particle projected vertically upwards with an initial velocity equal to the initial velocity of projection of a projectile.

104. *To find the greatest height attained.*

The projectile continues its upward motion until the vertical velocity becomes zero. Therefore, if H be the greatest height attained,

$$0 = u^2 \sin^2 \alpha - 2gH$$

$$\therefore \qquad H = \frac{u^2 \sin^2 \alpha}{2g}.$$

105. *To find the time to the highest point.*

At the highest point the vertical component of the velocity of the particle is zero.

Therefore if T be the time to the highest point,

$$0 = u \sin \alpha - gT$$

$$\therefore \qquad T = \frac{u \sin \alpha}{g}.$$

106. *To find the time of flight.*

Let t be the time of flight of the projectile, so that when the particle is again on the horizontal plane through the point of projection, the vertical distance traversed by it in time t is zero.

$$\therefore \qquad 0 = u \sin \alpha \, t - \tfrac{1}{2} gt^2$$

Hence $\quad t = 0 \quad$ or $\quad t = \dfrac{2u \sin \alpha}{g}$.

The first solution refers to the instant of projection.

$$\therefore \qquad \text{The time of flight} = \frac{2u \sin \alpha}{g}.$$

Also, the time of flight = twice the time to the highest point. Hence, the time from the point of projection to the highest point is the same as the time from the highest point back to the horizontal plane through the point of projection.

107. *To find the range on the horizontal plane.*

During the motion the horizontal velocity remains contant and is equal to $u \cos \alpha$.

$\therefore$ R=horizontal distance described in time t, (t being the time of flight)

$$R = u \cos \alpha . \; t = \frac{2u^2 \sin \alpha \cos \alpha}{g} = \frac{u^2 \sin 2\alpha}{g}.$$

The value of R remains unchanged when α is replaced by $\frac{\pi}{2} - \alpha$. Hence for a given velocity of projection there are two directions of projection for which the range on the horizontal plane through the point of projection is same.

Again, for a given value of u, the horizontal range is maximum, if, $\sin 2\alpha$ is maximum. The maximum value of $\sin 2\alpha$ is 1. Therefore $2\alpha = 90°$ and $\alpha = 45°$.

The above two directions of projection, α and $\frac{\pi}{2} - \alpha$ are equally inclined to the horizontal and vertical respectively and are therefore equally inclined to the direction of maximum range, which makes an angle 45° with the horizontal.

$$\therefore \quad R\,(\max) = \frac{u^2}{g}.$$

108. The path of projectile in vacuo is a parabola.

1st Method. Let u be the initial velocity and α the angle of projection, P the point of projection, A the highest point, PP′ the horizontal range and AM the perpendicular on PP′.

Then, by Art. 104,

$$AM = \frac{u^2 \sin^2 \alpha}{2g} \qquad \text{...........(1)}$$

and PM = the horizontal distance described in time $\frac{u \sin \alpha}{g}$.

= horizontal velocity × time

$$= \frac{u^2 \sin \alpha \cos \alpha}{g} \qquad \text{...........(2)}$$

Let Q be the position of the projectile at any time t, and let QL and QN be the perpendiculars on PP′ and AM respectively.

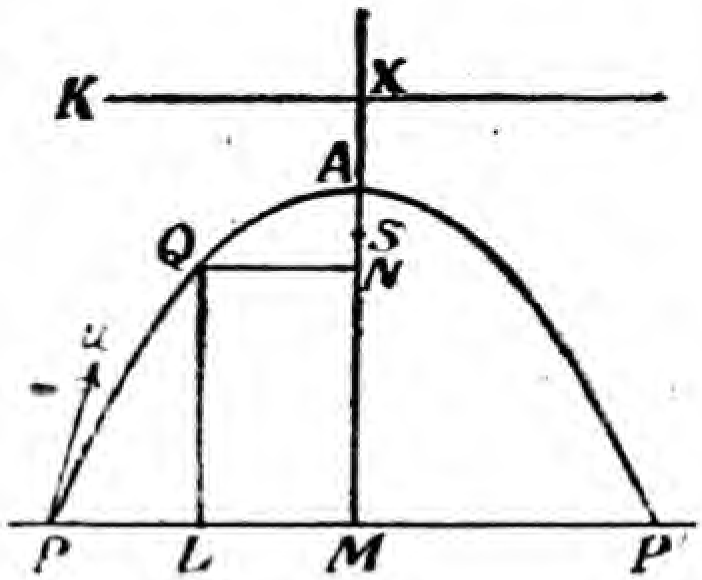


Fig. 54

Then QL = vertical distance described in time t

$$= u \sin \alpha\, t - \tfrac{1}{2} g t^2 \quad \ldots\ldots(3)$$

and
$$PL = u \cos \alpha\, t. \quad \ldots\ldots(4)$$

From (1) and (3),

$$AN = AM - NM = AM - QL$$

$$= \frac{u^2 \sin^2 \alpha}{2g} - (u \sin \alpha . t - \tfrac{1}{2} g t^2$$

$$= \frac{g}{2}\left(\frac{u \sin \alpha}{g} - t\right)^2$$

From (2) and (4),

$$QN = PM - PL = \frac{u^2 \sin \alpha \cos \alpha}{g} - u \cos \alpha\, t$$

$$= u \cos \alpha \left(\frac{u \sin \alpha}{g} - t\right)$$

$$QN^2 = u^2 \cos^2 \alpha \left(\frac{u \sin \alpha}{g} - t\right)^2$$

$$= u^2 \cos^2 \alpha \;.\; \frac{2AN}{g}$$

$$= \frac{2 u^2 \cos^2 \alpha}{g} \;.\; AN$$

$$= 4\, AS.\, AN, \text{ where we take } AS = \frac{u^2 \cos^2 \alpha}{2g}.$$

But this is the fundamental property of parabola.

Hence Q lies on a parabola whose axis is the vertical line AM, whose vertex is A, and whose latus rectum

$$= 4.AS = \frac{2 u^2 \cos^2 \alpha}{g}$$

Cor. I. The height of focus S above the horizontal line through P=SM.

$$SM = AM - AS$$

$$= \frac{u^2 \sin^2 \alpha}{2g} - \frac{u^2 \cos^2 \alpha}{2g} = -\frac{u^2}{2g} \cos 2\alpha$$

Hence, if α be less than 45°, this distance is negative and the focus of the path is then situated below the horizontal line through the point of projection.

Cor. II. The height of directrix XK above the horizontal through $P = AM + AX = AM + AS$

$$= \frac{u^2 \sin^2 \alpha}{2g} + \frac{u^2 \cos^2 \alpha}{2g} = \frac{u^2}{2g}.$$

Also the focal distance of the point of projection

$$SP = \sqrt{PM^2 + SM^2}$$

$$= \sqrt{\frac{u^4 \sin^2 2\alpha}{4g^2} + \frac{u^4 \cos^2 2\alpha}{4g^2}}$$

$$= \frac{u^2}{2g}.$$

or using the property of parabola,

SP=distance of P from the directrix

$$= XM = \frac{u^2}{2g}.$$

Cor. III. It follows from Cor. 2, that if a particle be projected from P with a given velocity u, inclined to the vertical, the focus of the parabolic path of the projectile lies on a circle, whose centre is P and radius $\frac{u^2}{2g}$; and the directrix is a horizontal tangent to this circle. *Hence the trajectories having the same point of projection and the same initial velocity have a common directrix.*

2nd Method. Referred to P as origin, PP′ as the X-axis and a line through P perpendicular to PP′ as the Y-axis, the co-ordinates of Q, after a time t' has elapsed, are

$$x = PL = u \cos \alpha . t \qquad \ldots\ldots(1)$$
$$y = QL = u \sin \alpha \; t - \tfrac{1}{2} g t^2 \ldots(2)$$

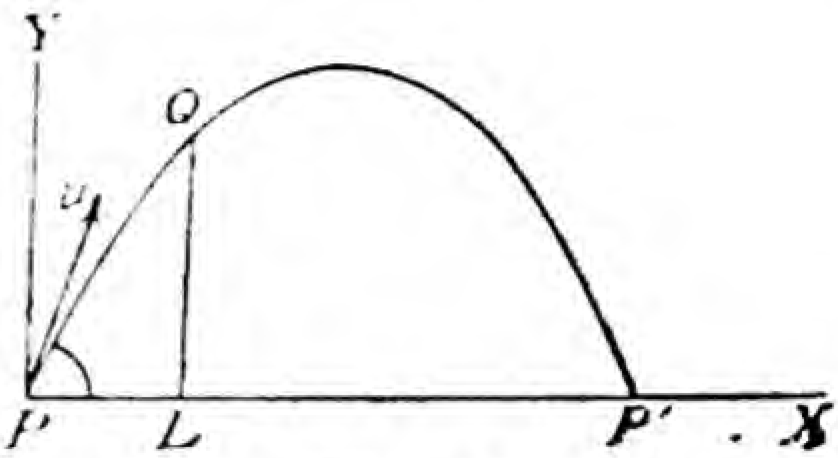


Fig. 55

From (1),

$$t = \frac{x}{u \cos \alpha} \qquad \ldots\ldots(3)$$

Putting this value of t in the relation (2) we get

$$y = \frac{u . \sin \alpha . x}{u \cos \alpha} - \tfrac{1}{2} g . \frac{x^2}{u^2 \cos^2 \alpha}$$

$$= x \tan \alpha - \frac{g x^2}{2 u^2 \cos^2 \alpha} \qquad \ldots\ldots\ldots(4)$$

This is the equation of the path of the particle and can be proved to be a parabola.

Equation (4) can be rewritten as

$$-\frac{2 u^2 \cos^2 \alpha}{g^2} y = x^2 - \frac{2 u^3 \sin \alpha \cos \alpha}{g} x \qquad \ldots\ldots\ldots(5)$$

Adding $\frac{u^4 \sin^2 \alpha \cos^2 \alpha}{g^2}$ to both sides of (5) in order to make the R. H. S. of the equation a perfect square, we get

$$-\frac{2 u^2 \cos^2 \alpha}{g} \left(y - \frac{u^2 \sin^2 \alpha}{2g} \right) = \left(x - \frac{u^2 \sin \alpha \cos \alpha}{g} \right)^2 \qquad \ldots\ldots\ldots(6)$$

Transforming the origin of co-ordinates to $\left(\frac{u^2 \sin \alpha \cos \alpha}{g}, \frac{u^2 \sin^2 \alpha}{2g} \right)$, that is, putting

$$x = X + \frac{u^2 \sin \alpha \cos \alpha}{g}$$

$$y = Y + \frac{u^2 \sin^2 \alpha}{2g}$$

in equation (6), we get

$$X^2 = -\frac{2u^2 \cos^2 \alpha}{g} Y$$

$$= -4\,a\,Y, \text{ where } a = \frac{u^2 \cos^2 \alpha}{2g} \quad \text{.........(7)}$$

(7) is the equation of a parabola, with concavity downwards. The co-ordinates of its vertex A are

$$\left(\frac{u^2 \sin \alpha \cos \alpha}{g}, \frac{u^2 \sin^2 \alpha}{2g} \right)$$

and its latus rectum is

$$\frac{2u^2 \cos^2 \alpha}{g}.$$

Ex. 1. Two seconds after its projection a projectile is travelling in a direction inclined at 30° to the horizon; after one more second it is travelling horizontally. Determine the magnitude and direction of its initial velocity.

Let u be the velocity of projection at an angle α to the horizon. By Art. 102,

$$\tan \theta = \frac{u \sin \alpha - g\,t}{u \cos \alpha}$$

After two seconds,

$$\therefore \quad \tan. 30° = \frac{u \sin \alpha - 2g}{u \cos \alpha} \quad \text{or} \quad \frac{1}{\sqrt{3}} u \cos \alpha = u \sin \alpha - 2g \text{...(1)}$$

Again after three seconds,

$$\tan 0° = \frac{u \sin \alpha - 3g}{u \cos \alpha}$$

i.e. $u \sin \alpha - 3g = 0$ ....(2)

whence $u \sin \alpha = 3g$ ....(3)

Substituting in (1), we get

$u \cos \alpha = g\sqrt{3}$ ....(4)

Squaring (3) and adding it to the square of (4),

$$u^2(\cos^2\alpha+\sin^2\alpha)=9g^2+3g^2$$

$$u^2=12\ g^2$$

or $$u=2g\ \sqrt{3}.$$

$$\tan\alpha=\frac{u\sin\alpha}{u\cos\alpha}=\sqrt{3}.$$

$\therefore$ $$\alpha=60^\circ.$$

Ex. 2. Determine the least velocity with which a ball can be thrown to reach the top of a cliff 128 ft. high and 128 $\sqrt{3}$ ft. away from the point of projection.

Suppose u to be the initial velocity and α the direction of projection. By Art. 108 (II) the equation of the path is

$$y=x\tan\alpha-\frac{g\ x^2}{2u^2\cos^2\alpha}$$

Putting $y=128$ and $x=128\sqrt{3}$, we get

$$128=128\ \sqrt{3}.\ \tan\alpha-\frac{32\times128^2\times3}{2\ u^2\cos^2\alpha}$$

$$u^2=\frac{16\times128\times3}{\cos\alpha\ [\sin\alpha\ .\ \sqrt{3}-\cos\alpha}$$

$$=\frac{16\times128\times3}{2\cos\alpha\sin(\alpha-30)}=\frac{16\times128\times3}{\sin(2\alpha-30)-\sin 30}\ \ldots(1)$$

u^2 is least, when the denominator of (1) is maximum, *i.e.*,

$$\sin(2\alpha-30)=1$$

or $$2\ \alpha-30^\circ=90^\circ$$

$$\alpha=60^\circ.$$

$$\therefore\ (U^2)\text{ least}=\frac{16\times128\times3}{1-\frac{1}{2}}$$

$$\therefore\ (U)\text{ least}=64\ \sqrt{3}\text{ ft./sec.}$$

Ex. 3. Supposing a gun elevated at 90° can send a shot to a height h feet, to keep in the air T seconds, prove that at an angle of elevation α the range on a horizontal plane will be $2\ h\sin 2\ \alpha$, with a time of flight T $\sin\ \alpha$ seconds; and the greatest height attained will be $h.\ \sin^2\alpha$.

The velocity with which the gun can fire a projectile

$$u=\sqrt{2\,g\,h}. \qquad \ldots\ldots\ldots(1)$$

Also $\quad 0= u\,\mathrm{T}-g\,\dfrac{\mathrm{T}}{2}$

or $\quad \mathrm{T} \;=\dfrac{2u}{g}=2\sqrt{\dfrac{2h}{g}}.$

By Art. 107, the range on the horizontal plane, when the projectile is fired with an initial velocity u at an angle α to the horizon, is

$$\frac{u^2 \sin 2\alpha}{g}=\frac{2\,gh \sin 2\,\alpha}{g}=2\,h \sin 2\,\alpha.$$

Again the time of flight $=\dfrac{\text{Horizontal range}}{\text{Horizontal Comp. of velocity}}$

$$=\frac{2\,h\,\sin 2\alpha}{u\cos\alpha}=\frac{4\,h\sin\alpha}{\sqrt{2\,g\,h}}$$

$$=2.\,\sin\alpha\sqrt{\frac{2\,h}{g}}$$

$$=\mathrm{T}\sin\alpha$$

Greatest height attained $=\dfrac{u^2\sin^2\alpha}{2\,g}=\dfrac{2\,g\,h\sin^2\alpha}{2\,g}$

$$=h\sin^2\alpha.$$

Examples XII

1. Show that, for a body projected at an angle of 30° to the horizon, the range is the same as for a body projected at an angle of 60° with the same velocity. Compare the greatest height attained in the two cases.

2. Slow that when a particle comes back to the horizontal plane from which it was projected, its horizontal velocity remains unchanged, but its vertical velocity is reversed.

3. A particle is projected from the top of a tower in a horizontal direction with a velocity of 6 ft per sec. and takes two seconds to fall to the ground. Find where it strikes the ground and where it was at the end of the first second.

4.. Find the velocity and the direction of projection of a shot which passes in a horizontal direction just over the top of a wall which is 50 yds. off and 75 feet high.

5. Two particles are projected in directions inclined at $30°$ and $60°$ to the horizon; if they reach the same height, show that their velocities of projection are in the ratio $\sqrt{3} : 1$.

6. A stone is thrown horizontally, with a velocity $\sqrt{2gh}$, from the top of a tower of height h. Find where it will strike the level ground through the foot of the tower. What will be its striking velocity?

7. Find the angle of projection when the range is equal to the distance through which the particle would have to fall in order to acquire a velocity equal to its velocity of projection.

8. A ball is thrown from a point 7 feet above level ground. It runs to a maximum height of 16 feet above the ground and strikes it at a horizontal distance of 105 feet from the point of projection. Find the velocity with which the ball is thrown and the angle at which it is thrown.

9. If two bodies, projected at the same instant from different points in the same horizontal plane, be at any moment at the same height above the plane, show that their heights are the same at any subsequent time.

10. A cricket ball thrown from a height of 6 ft. at an angle of $30°$ with the horizon with a speed of 60 ft./sec., is caught by another fieldsman at a height of 2 feet from the ground. How far apart were the two men?

11. The greatest height to which a man can throw a stone is h. What is the greatest distance to which he can throw it, and in that case how long is it in the air?

12. Smooth heavy particles are let fall simultaneously down the chords of a vertical circle from its highest point. Show that they all reach the circumference again at the same instant, and that their subsequent parabolic paths have the same directrix. [For the first part see Art. 43.]

13. An airman flying horizontally at 120 m.p.h., at a height of 1600 feet wants to drop a bomb to hit a target on the ground. How far away from the target is the point vertically below him on the ground at the instant when he drops the bomb?

109. *Velocity at any point of the trajectory is equal to that due to a free fall from the directrix.*

In the figure of Art 108, if v is the velocity at Q, we have by Art 103,

$$v^2 = u^2 - 2gh \qquad \text{where } h = QL$$

$$\text{Also } XM = \frac{u^2}{2g}; \qquad \text{(Art. 108. Cor. II)}$$

$$\therefore \quad \text{Depth of Q below the directrix} = XN = \frac{u^2}{2g} - h.$$

Therefore the velocity acquired by a particle after falling from rest a distance XN is

$$V^2 = 2g\left(\frac{u^2}{2g} - h\right)$$

$$= u^2 - 2gh.$$

$$\therefore \quad v = V.$$

110. Range on an inclined plane. Let u be the velocity of projection, α the angle of projection and PQ an inclined plane making an angle β with the horizontal. If the projectile strikes the inclined plane at Q, PQ is the range on the inclined plane. Draw QN perpendicular to the horizontal plane through P.

1st Method. The initial component of the velocity perpendicular to PQ is $u \sin(\alpha - \beta)$, and the acceleration in the same direction is $-g \cos \beta$. If T is the time which the particle takes in going from P to Q, then during this time the space described in a direction perpendicular to PQ is zero.

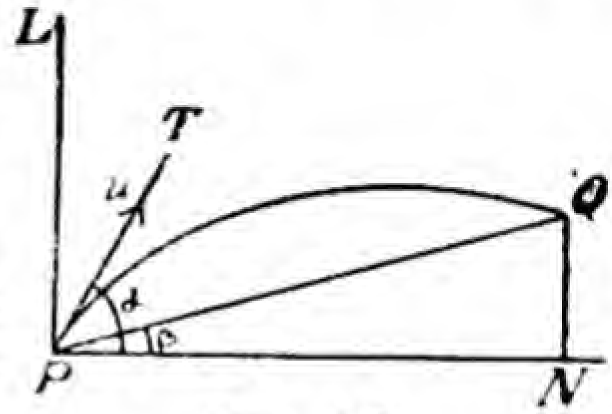


Fig. 56

Hence $O = u \sin(\alpha - \beta).$

$$T - \tfrac{1}{2} g \cos \beta . T^2$$

$$\therefore \quad T = \frac{2u \sin(\alpha - \beta)}{g \cos \beta}. \qquad \ldots\ldots\ldots\ldots \quad (1)$$

During this time the horizontal velocity of the projectile remains unchanged and is equal to $u \cos \alpha$.

$$\therefore \quad PN = u \cos \alpha . \; T$$

$$\therefore \quad PQ = \frac{PN}{\cos \beta} = \frac{2u^2 \cos \alpha \sin(\alpha - \beta)}{\cos^2 \beta}.$$

2nd Method. We proceed to find out the horizontal and vertical distances travelled by the projectile.

$$PN = u \cos \alpha \, t$$

$$QN = u \sin \alpha . \; t - \tfrac{1}{2} g t^2$$

Now $\tan \beta = \dfrac{QN}{PN} = \dfrac{u \sin \alpha . t - \frac{1}{2} g t^2}{u \cos \alpha \, t} = \tan \alpha - \dfrac{gt}{2u \cos \alpha}$

$$\therefore \quad t=\frac{2u\cos\alpha}{g}(\tan\alpha-\tan\beta)=\frac{2u\sin(\alpha-\beta)}{g\cos\beta}.$$

$$PN=u\cos\alpha . t$$

and $$PQ=\frac{PN}{\cos\beta}=\frac{2u^2}{g}\,\frac{\cos\alpha\sin(\alpha-\beta)}{\cos^2\beta}.$$

3rd Method. With P as origin, PN as X-axis and PL as Y-axis the equation of the path is

$$y=x\tan\alpha-\frac{g.\,x^2}{2u^2\cos^2\alpha} \qquad \text{............(2)}$$

If PQ$=r$, the co-ordinates of Q are $(r\cos\beta, r\sin\beta)$. Substituting the co-ordinates of Q in (1) and after a little simplification we get,

$$r=\frac{2u^2}{g}\,\frac{\cos\alpha\sin(\alpha-\beta)}{\cos^2\beta}$$

111. Maximum Range. *To find the direction of projection which gives the maximum range on an inclined plane, and to show that for any given range there are two directions of projection, which are equally inclined to the direction for maximum range.*

From the preceding article the range

$$=\frac{2u^2\cos\alpha\sin(\alpha-\beta)}{g\cos^2\beta}=\frac{u^2}{g\cos^2\beta}\left\{\sin(2\alpha-\beta)-\sin\beta\right\}$$

............(i)

Now u and β are given; hence the range is maximum when

$\sin(2\alpha-\beta)$ is greatest, or when $2\alpha-\beta=\frac{\pi}{2}$.

In this case $\alpha-\beta=\frac{\pi}{2}-\alpha$, *i.e.*, the angles TPQ and LPT are equal.

Hence the direction for maximum range bisects the angle between the vertical and the inclined plane.

Also the maximum range

$$=\frac{u^2}{g\cos^2\beta}(1-\sin\beta)=\frac{u^2}{g(1+\sin\beta)}$$

Again, the range with an angle of elevation α_1 is, by (i) the same as that with elevation α, if

$$\sin(2\alpha_1-\beta)=\sin(2\alpha-\beta)$$

i.e., if
$$2\alpha_1-\beta=\pi-(2\alpha-\beta)$$

or
$$\alpha_1=\frac{\pi}{2}+\beta-\alpha$$

or
$$\alpha_1-\left(\frac{\pi}{4}+\frac{\beta}{2}\right)=\left(\frac{\pi}{4}+\frac{\beta}{2}\right)-\alpha.$$

But $\frac{\pi}{4}+\frac{\beta}{2}$ is the elevation which gives the greatest range.

Hence for any given range on an inclined plane there are two angles of projection, the two corresponding directions of projection being equally inclined to that for the maximum range on the plane.

Ex. 1. A particle is projected with a velocity of 64 ft. per second at an elevation of 60° (*i*) up (*ii*) down an inclined plane of inclination 30° to the horizon, and passing through the point of projection ; find in each case the range on the plane and the time of flight.

(*i*) By Art. 110,

$$\text{Range}=\frac{2\,(64)^2\cos 60°\sin(60°-30°)}{32\times\cos^2 30°}$$

$$=\frac{2\times 64\times 64\times 4}{32\times 3\times 4}=\frac{256}{3}=85\frac{1}{3}\text{ ft.}$$

Again, by Art. 110, [1st method (*i*)].

$$\text{Time}=\frac{2\times 64\sin(60°-30°)}{32\times\cos 30°}$$

$$=\frac{4}{\sqrt{3}}\text{ seconds.}$$

(*ii*) When the particle is projected down the plane, we replace β in Art. 110 by $(-\beta)$, and get

$$\text{Range}=\frac{2\times(64)^2\cos 60°\sin(60°+30°)}{32\cos^2 30°}$$

$$=\frac{4\times 64\times 4}{3\times 2}=170\frac{2}{3}\text{ ft.}$$

And Time$=\dfrac{2\times(64)\sin(60^\circ+30^\circ)}{32\times\cos 30^\circ}=\dfrac{8}{\sqrt{3}}$ seconds.

Ex. 2. A particle is projected at an angle α with the horizontal from the foot of a plane, whose inclination to the horizon is β; show that it will strike the plane at right angles, if $\cot\beta=2\tan(\alpha-\beta)$.

Let u be the velocity of projection, so that $u\cos(\alpha-\beta)$ and $u\sin(\alpha-\beta)$ are the initial velocities respectively parallel and perpendicular to the inclined plane.

The components of acceleration along and perpendicular to the inclined plane are $-g\sin\beta$ and $-g\cos\beta$.

By Art 110, the time T, that elapses before the particle reaches the plane again is $\dfrac{2v\sin(\alpha-\beta)}{g\cos\beta}$.

If the direction of motion at the instant when the particle hits the plane be perpendicular to the plane, then the velocity at that instant parallel to the inclined plane must be zero.

Hence $u\cos(\alpha-\beta)-g\sin\beta\ \mathrm{T}=0$,

$$\therefore\quad \frac{u\cos(\alpha-\beta)}{g\sin\beta}=\mathrm{T}=\frac{2u\sin(\alpha-\beta)}{g\cos\beta}.$$

$$\therefore\quad \cot\beta=2\tan(\alpha-\beta).$$

Ex. 3. The angular elevation of an enemy's position on a hill h feet high is β; show that, in order to shell it, the initial velocity of the projectile must not be less than

$$\sqrt{gh(1+\operatorname{cosec}\beta)}.$$

Suppose the projectile is projected with a velocity u making an angle α with the horizontal plane. The velocity of projection will be least, if the range on the plane joining the point of projection and the enemy's position corresponding to u is maximum.

Using Art 110, we have

$$h\operatorname{cosec}\beta=\frac{2u^2\cos\alpha\sin(\alpha-\beta)}{g\cos^2\beta}$$

or $$u^2=\frac{gh\cos^2\beta\operatorname{cosec}\beta}{\sin(2\alpha-\beta)-\sin\beta}.$$

u is least when $\sin(2\alpha-\beta)$ is greatest, *i.e.*, $2\alpha-\beta=\frac{\pi}{2}$.

$$\therefore\ (u)\ \text{least}=\sqrt{\frac{gh\cos^2\beta\operatorname{cosec}\beta}{1-\sin\beta}}=\sqrt{gh(1+\sin\beta)\operatorname{cosec}\beta}$$
$$=\sqrt{gh(1+\operatorname{cosec}\beta)}.$$

Ex. 4. If t be the time in which a projectile reaches a point P of its path, and t' be the time trom P till it strikes the horizontal plane through the point of projection, show that the height of P above the plane is $\frac{1}{2}g\,t\,t'$.

Let u be the velocity of projection making an angle α with the horizontal plane.

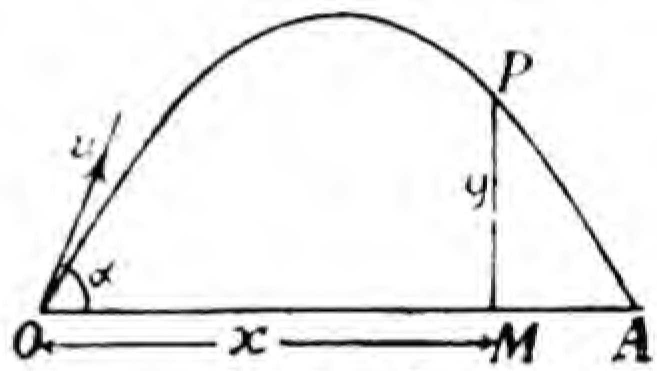


Fig. 57

Referred to O as origin, let (x, y) be the co-ordinates of P. Then

$x=u\cos\alpha.\,t$ .... (1)

$y=u\sin\alpha.\,t-\frac{1}{2}gt^2$ .... (2)

Now OA = Range on the horizontal plane

$$=\frac{u^2\sin 2\alpha}{g}.$$

$$\therefore\qquad AM=\frac{u^2\sin 2\alpha}{g}-x.$$

Since the projectile takes t' time in moving from P to A,

$$u\cos\alpha.\ t'=\frac{u^2\sin 2\alpha}{g}-x \qquad ....(3)$$

From (1) and (3),

$$u\cos\alpha\,(t+t')=\frac{u^2\sin 2\alpha}{g}.$$

or
$$u=\frac{g\,(t+t')}{2\sin\alpha} \qquad ....(4)$$

Substituting the value of u in (2), we get

$$y=\frac{g}{2\sin\alpha}(t+t')\sin\alpha.\ t-\frac{g}{2}t^2$$
$$=\tfrac{1}{2}g\,t\,t'.$$

Examples XIII

1. A gun pointed at an inclination θ to the horizontal is mounted on a carriage which can run on a horizontal rail. If the shot leaves the gun with a relative velocity v, the gun recoiling along the rail with velocity u, show how to find the range on a horizontal plane through the point of projection. By how much is the range shortened by the recoil, if $v=1000$ ft./sec. and $u=10$ ft./sec. $\theta=45°$?

2. Water is issuing from a fire-hose nozzle with a speed of 120 feet per second. The jet is to pass through a window which is distant 35 feet vertically and 3 feet horizontally from the nozzle. What must be the inclination of the nozzle if the jet when it reaches the window is (1) to be rising, (2) to be falling.

3. A particle slides down a smooth straight tube and then falls freely under the action of gravity; prove that the directrix of its parabolic path passes through the upper end of the tube.

4. Two men 4 ft. 7 inches apart in a railway carriage which is moving uniformly at the rate of 45 m. p. h. throw a ball from one to the other, projecting it so that its time of flight is $\frac{1}{8}$ second. Show that the ball describes a parabola in space; and taking its path in the carriage to be perpendicular to the rails, find its range in space.

5. A particle is projected from a point on an inclined plane of angle β with velocity u at an angle α to the horizon so that its path is in the same plane as the line of the greatest slope in the plane. With what velocity must I move on the plane so as to be always vertically under the particle?

6. A particle is projected horizontally from the top of a tower. Show that the path will be a parabola. Find also the depth of the focus of the parabola below the point of projection.

7. If the horizontal velocity of a bullet be 2000 feet per second, find the slope at which the gun should be aimed, if the bullet is to hit a mark 6 feet above the muzzle at a range of 500 yds.

8. If a particle projected with an initial velocity u, has a range R on a horizontal plane, show the time of flight T, and the greatest height H can be obtained from the equation

$$g^2T^4-4T^2u^2+4R^2=0 \text{ and } 16\,gH^2-8u^2H+gR^2=0.$$

9. There are two poles of heights h and H feet at a distance d ft. apart. From what point on the ground must a stone be projected if it is just to go over the first and lower pole and just to reach the top of the second and higher pole?

10. From a point in a given inclined plane two bodies are projected with the same velocity in the same vertical plane at right angles to one another; show that the difference of their ranges is constant.

11. A particle, projected with a velocity u, strikes at right angles a plane through the point of projection inclined at an β to the horizon.

Show that the height of the point struck above the horizontal plane through the point of projection is $\frac{2u^2}{g}\frac{\sin^2\beta}{1+3\sin^2\beta}$, that the time of flight is $\frac{2u}{g\sqrt{1+3\sin^2\beta}}$, and the range on a horizontal plane through the point of projection is

$$\frac{u^2\sin 2\beta}{g}\frac{1+\sin^2\beta}{1+3\sin^2\beta}$$

12. A particle is projected with velocity $2\sqrt{ag}$ so that it just clears two walls of equal height a, which are at a distance $2a$ from each other. Show that the latus rectum of the path is $2a$, and that the time of passing between the walls is $2\sqrt{\frac{a}{g}}$.

13. If at any point of a parabolic path the velocity be u and the inclination to the horizon be θ, show that the particle is moving at right angles to its former direction after a time $\frac{u}{g\sin\theta}$.

14. Show that for a given velocity of projection the maximum range down a plane of inclination α is greater than up the plane in the ratio $\frac{1+\sin\alpha}{1-\sin\alpha}$.

15. A stone is projected at an angle α with the horizon from the bas of a plane of inclination β to the horizon, (the trajectory lying in the vertical plane containing the line of greatest slope through the point). Show that, if the elevation above the horizon of the point of the pat most distant from the inclined plane is θ, then

$$2\tan\theta=\tan\alpha+\tan\beta.$$

CHAPTER IX

IMPACT OF ELASTIC BODIES

112. Elasticity. Take a piece of steel wire and tie one end of it to the ceiling and suspend a small weight from the other end. The wire stretches in length a little, but as soon as the weight is removed it contracts to its original length. The deformation produced in the length of the wire is temporary and the wire regains its original length due to a certain property of the material, called the *Elasticity*.

Again, take a number of similar spherical balls, say of iron, wood, glass, ivory etc., and drop them on a marble floor from the same height. While approaching the floor, all the balls strike it with the same velocity, but it is found that the balls *rise to different heights after the impact.*

Again, the same ball of iron rises to different heights after striking a floor of marble and a floor of wood, although it was dropped from the same height.

The reason is precisely as follows : the impulsive action between the ball and the floor during the short period for which the impact lasts, produced a momentary compression of their surfaces of contact but immediately afterwards, the internal forces come into play within each body, as a result of which the ball and the floor regain their original shape. These forces produce an upward momentum in the ball causing it to move upwards, and being of different magnitude in different materials the ball rises to different heights.

Definition. The property of material bodies by virtue of which they can be compressed and after compression they recover or tend to recover their original shape is called *Elasticity.*

113. Direct and Oblique Impact. When two bodies collide with each other, they are said to *impinge* on one another. The phenomena of colliding is called an impact.

An impact between two bodies is said to be direct if the direction of motion of each, just before impact, is along the common normal at the point of contact.

Two bodies are said to impinge obliquely, when the velocity of at least one of them is not along the common normal at the point of contact.

The direction of the common normal is called the *line of impact*. In the case of two spheres the line joining their centres is the common normal.

114. Laws of Impact. We enunciate below the laws when the impact is direct.

1. The Law of Conservation of Momentum. Let us suppose that two spheres have masses m_1 and m_2 and initial velocities u_1 and u_2, respectively along the same straight line. After impact the velocities become v_1 and v_2, respectively. Let the time of impact be t and let the force with which m_2 cats on m_1 during this time be F. The impulse of the force is equal to the change in momentum of the body on which the force acts. We have

$$R = Ft = m_1 (u_1 - v_1)$$

Now from Newton's Third Law of Action and Reaction, it follows that the force with which m_1 acts on m_2 during the impact is $-F$. We have, therefore,

$$-R = -Ft = m_2 \ (u_2 - v_2)$$

From these two equations, we have

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2.$$

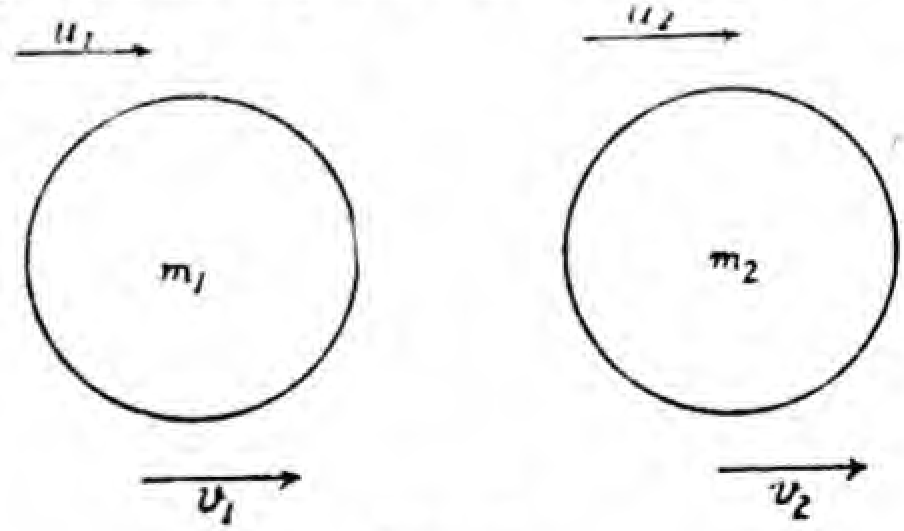


Fig. 58

Hence the sum of momenta of the bodies along the normal remains unaltered by impact between them.

2. Newton's Experimental law. Newton found by experiments that when two bodies collide directly, their relative velocity after impact is in a constant ratio to their relative

velocity before impact, the latter being reckoned in the same direction as the former. This constant does not depend on the masses of the impinging bodies but depends only on the material of the bodies. It is denoted by e and is called the co-efficient (or modulus) of restitution. Hence

$$v_1 - v_2 = -e(u_1 - u_2)$$

This result is sometimes expressed in another form, that is, the velocity of separation $= -e$ (velocity of approach),

115. Value of e. More careful experiments have revealed that Newton's law is only approximately true. For large velocities of approach the value of e decreases as the velocity of approach increases. The effect is more noticeable for large velocities of approach ; e has different values for different pairs of substances. The maximum and minimum values for e are 1 and 0 respectively. Bodies for which $e=1$ are called *perfectly elastic bodies*, and the bodies for which $e=0$ are called *perfectly inelastic bodies*.

Ex. 1. A sphere impinges directly on an equal sphere at rest. If the co-efficient of restitution be e, show that their velocities after the impact are as $1-e : 1+e$.

Let u_1 and v_1 be the velocities of the impinging ball before and after the impact ; let v_2 be the velocity which is imparted to the second ball as result of impact, then, by the Law of Conservation of Momentum,

$$mu_1 = mv_1 + mv_2$$

or

$$u_1 = v_1 + v_2$$

Again, $-e$.(The velocity of approach) = The velocity of separation.

$$-eu_1 = v_1 - v_2$$

Therefore

$$v_1 = u_1 (1-e)$$

and

$$v_2 = u_1 (1+e)$$

Hence,

$$v_1 : v_2 :: 1-e : 1+e$$

116. Direct Impact of two smooth solid spheres. A smooth sphere, of mass m_1, impinges directly with velocity u_1 on another smooth sphere, of mass m_2, moving in the same direction with velocity u_2. If the co-efficient of restitution be e, to find their velocity after the impact.

Let v_1 and v_2 be the veloeities of the two spheres after the impact.

By Newton's experimental law, we have

$$v_1 - v_2 = -e(u_1 - u_2) \qquad \ldots(1)$$

By the Law of Conservation of Momentum, we have

$$m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2 \qquad \ldots(2)$$

Multiply (1) by m_2 and add to (2)

$$(m_1 + m_2)v_1 = (m_1 - em_2)u_1 + m_2(1+e)u_2$$

Again multiplying (1) by m_1 and subtracting from (2) we have

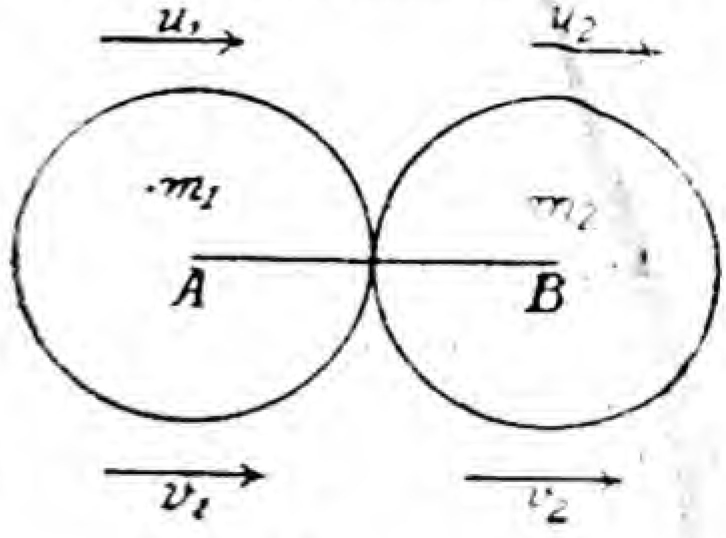


Fig. 59

$$(m_1 + m_2)v_2 = m_1(1+e)u_1 + (m_2 - em_1)u_2$$

These two equations give the velocities after the impact. Also the impulse of the blow on the ball m_1

$$= \text{the change produced in its momentum.}$$

$$= m_1 \quad (u_1 - v_1) = \frac{m_1 m_2}{m_1 + m_2} \ (1+e)(u_1 - u_2)$$

Cor. If we put $m_1 = m_2$ and $e = 1$, we have

$$v_1 = u_2 \text{ and } v_2 = u_1$$

[Hence *if two equal perfectly elastic balls impinge directly the yinterchange their velocities.*

Ex. 1. A ball of mass 8 pounds moving with velocity 4 ft./sec. overtakes a ball of mass 10 pounds, moving with velocity 2 ft./sec. in the same direction. If $e = \frac{1}{2}$, find the velocities of the balls after the impact. Calculate also the impulse of the blow on the two spheres separately.

Suppose their velocities after impact are v_1 and v_2. By Newton's Law, we have,

$$v_1 - v_2 = -\tfrac{1}{2}(4-2) \qquad \ldots(1)$$

By the Law of Conservation of Momentum,

$$8 \times 4 + 10 \times 2 = 8v_1 + 10v_2 \qquad \ldots(2)$$

(1) becomes $\quad v_1 - v_2 = -1$

(2) becomes $\quad 4v_1 + 5v_2 = 26$

Hence, $v_1 = 2\frac{1}{3}$ ft./sec. ; $v_2 = 3\frac{1}{3}$ ft./sec.

Now the impulse of blow on the 8-pound sphere

$= \text{change of momentum in the 8-pound sphere}$

$= 8\,(4 - 2\frac{1}{3})$ units of momentum

$= 13\frac{1}{3}$ units of momentum.

The impulse of the blow on the 10 lb.-sphere will also be $13\frac{1}{3}$ units of momentum, but in the opposite direction.

Ex. 2. Three perfectly elastic balls of masses m, $2m$ and $3m$ are placed in a straight line. The first impinges directly on the second with a velocity u and then the second impinges on the third. Find the velocity of the third ball after impact.

If the masses of the second and third balls are m_1 and m_2, show that the third ball will move after impact with velocity u, if $(m+m_1)(m_1+m_2)=4mm_1$.

(1) Let u and v be the velocities of the first ball before and after the impact. Let v_1 be the velocity of the second ball after it has collided with the first ball.

By the Law of Conservation of Momentum,

$$mu = mv + 2mv_1 \qquad \ldots(1)$$

or

$$u = v + 2v_1 \qquad \ldots(2)$$

Again,

The velocity of separation $= -e \times$ The velocity of approach.

$$v - v_1 = -u \qquad \ldots(3)$$

Hence $\quad v_1 = \frac{2}{3}u.$

Now, let v_2 be the velocity of the second ball after its impact with the third; and also let v_3 be the velocity of the third ball after the impact, then

$$2m.\,\tfrac{2}{3}u = 2m\,v_2 + 3m\,v_3$$

or

$$\tfrac{4}{3}u = 2v_2 + 3v_3 \qquad \ldots(4)$$

And

$$v_2 - v_3 = -\tfrac{2}{3}u \qquad \ldots(5)$$

From (4) and (5) $\quad v_3 = \frac{8}{15}u.$

(2) With the same meaning for u, v_1, v_2 and v_3; but taking the masses of the second and third balls as m_1 and m_2, we get

$$mu = mv + m_1v_1 \qquad \ldots(6)$$

$$v - v_1 = -u \qquad \ldots(7)$$

$$m_1v_1 = m_1v_2 + m_2v_3 \qquad \ldots(8)$$

$$v_2 - v_3 = -v_1 \qquad \ldots(9)$$

From (6) and (7), we get,

$$v_1 = \frac{2mu}{m+m_1} \qquad \ldots(10)$$

From (8) and (9), we get,

$$v_3 = \frac{2m_1 v_1}{(m_1+m_2)} \qquad \ldots(11)$$

From (10) and (11), we have,

$$v_3 = \frac{4mm_1 . u}{(m+m_1)(m_1+m_2)}.$$

If $v_3 = u$, we get, $(m+m_1)(m_1+m_2) = 4mm_1$.

Ex. 3. A particle falls from a height h, upon a fixed horizontal plane; if e be the coefficient of restitution, show that the whole distance described by the particle before it has finished rebounding is $\frac{1+e^2}{1-e^2} h$, and the time that elapses is $\sqrt{\frac{2h}{g}} \left[\frac{1+e}{1-e}\right]$.

Let v be the velocity of the particle when it first hits the plane, so that $u^2 = 2gh$.

The particle then rebounds with a velocity eu. The velocity, when it again hits the plane is eu and the velocity after the second rebound is e^2u.

Similarly the velocity after the third, fourth,......rebounds is e^3u, e^4u,............

The height to which the particle ascends after the first, second, third,......rebounds are $\frac{e^2u^2}{2g}$, $\frac{(e^2u)^2}{2g}$, $\frac{(e^3u)^2}{2g}$,............i.e., e^2h, e^4h, e^6h,............

Hence the whole space described

$$= h + 2(e^2h + e^4h + e^6h + \ldots\ldots \text{ad inf.})$$

$$= h + 2h\frac{e^2}{1-e^2}$$

$$= h\,\frac{1+e^2}{1-e^2}.$$

The times of ascending after the impacts are the times in which the velocities eu, e^2u, e^3u,......are destroyed by gravity.

Hence these times are $\frac{eu}{g}$, $\frac{e^2u}{g}$, $\frac{e^3u}{g}$,......

i.e., $e\sqrt{\frac{2h}{g}}$, $e^2\sqrt{\frac{2h}{g}}$,......

Hence the whole time during which the particle is in motion

$$=\sqrt{\frac{2h}{g}}+2\sqrt{\frac{2h}{g}}[e+e^2+e^3+\text{........ad inf.}]$$

$$=\sqrt{\frac{2h}{g}}\left[1+\frac{2e}{1-e}\right]=\sqrt{\frac{2h}{g}}\left[\frac{1+e}{1-e}\right].$$

In theory we have an infinite number of rebounds taking place in a finite time; in practice after a few rebounds the velocity of the ball becomes destroyed.

117. Action between two elastic bodies during their Collision. When two elastic bodies collide, the action between them during their collision may be divided into parts: force of compression and the force of restitution. In the case of two railway carriages as soon as the carriages collide the buffers are compressed and then they expand and after some time regain their original uncompressed lengths.

Similarly, if an elastic ball, covered with fine coloured powder, is dropped on a ground floor, we find that the powder is removed from a circular area of the surface of the ball. This shows that the contact between the ball and the ground does not take place at a single point, but over a certain area and this is possible only when the ball is compressed at the point of contact. Hence in the earlier stages the ball is compressed. But soon after it rebounds and moves upwards. This shows that during the latter stages the ball expands and recovers its original shape.

The period of impact can thus be divided into two parts: the period of compression and the period of restitution. During the period of compression the forces of compression are acting on the colliding bodies and during the period of restitution the forces of restitution come into play, which help the body to regain its original shape.

In the case of perfectly inelastic bodies, there is no force of restitution and the bodies are permanently deformed.

The force between the bodies at the commencement of the impact is zero. It attains a maximum value at some instant during which the impact lasts, and is again zero when the bodies are on the point of separation. By Newton's Third Law, the force at each instant must be same in magnitude but opposite in direction for each body; hence the impulses on the two bodies are equal in magnitude but opposite in direction.

118. The Impulses of Compression and Restitution Let I denote the total impulse on one body, and let I_c and I_r be the impulses of compression and restitution.

Then

$$I = I_c + I_r$$

From Art. 116,

$$I = I_c + I_r = \frac{m_1 m_2 (1+e)(u_1 - u_2)}{m_1 + m_2}$$

But $\quad I_r = 0 \quad \text{if } e = 0$

Therefore putting $e = 0$, we get

$$I_c = \frac{m_1 m_2 (u_1 - u_2)}{m_1 + m_2}$$

Hence $\quad I_r = \dfrac{e.m_1 m_2 (u_1 - u_2)}{m_1 + m_2}$

Therefore, $I_r = e I_c$

Hence a quantitative *definition of e* (co-efficient of restitution) is furnished by the above equation.

119. Loss of Kinetic Energy by impact. Two spheres of given masses moving with given velocities impinge; to show that there is a loss of kinetic energy and to find the amount.

Let m_1 and m_2 be masses of the spheres. Let u_1, u_2 and v_1, v_2 be their velocities before and after the impact respectively.

Let e be the coefficient of restitution.

In the case of direct impact

$$m_1v_1+m_2v_2=m_1u_1+m_2u_2 \qquad ...(1)$$
$$v_1-v_2=-e(u_1-u_2) \qquad ...(2)$$

To the square of (1) add the square of (2) multiplied by m_1m_2, we get,

$$(m_1^2+m_1m_2)\,v_1^2+(m_2^2+m_1m_2)v_2^2$$
$$=(m_1u_1+m_2u_2)^2+e^2m_1m_2(u_1-u_2)^2$$

or
$$(m_1+m_2)(m_1v_1^2+m_2v_2^2)$$
$$=(m_1u_1+m_2u_2)^2+m_1m_2(u_1-u_2)^2-(1-e^2)m_1m_2(u_1-u_2)^2$$
$$=(m_1+m_2)(m_1u_1^2+m_2u_2^2)-(1-e^2)m_1m_2(u_1-u_2)^2$$

Dividing throughout by $2(m_1+m_2)$

$$\tfrac{1}{2}\,m_1v_1^2+\tfrac{1}{2}\,m_2v_2^2$$
$$=\tfrac{1}{2}\,m_1u_1^2+\tfrac{1}{2}\,m_2u_2^2-\frac{1-e^2}{2}\,\frac{m_1m_2}{m_1+m_2}\,(u_1-u_2)^2 \quad ...(3)$$

$$\text{Final K.E.}-\text{Initial K.E.}=-\frac{1-e^2}{2}\,\frac{m_1m_2}{m_1+m_2}\,(u_1-u_2)^2$$

Hence the loss of K.E. is

$$\frac{1-e^2}{2}\,\frac{m_1m_2}{m_1+m_2}\,(u_1-u_2)^2$$

When the balls are perfectly elastic, $e=1$, and there is no loss of K.E.

Examples XIV

1. A heavy elastic ball drops from the ceiling of a room and after rebounding twice from the floor reaches a height equal to one half that of the ceiling. Find the coefficient of restitution.

2. A sphere of 2 pounds moving with a velocity of 4 ft /sec. impinges directly on another sphere of mass 4 lbs. at rest. If the first sphere comes to rest after impact, find the coefficient of restitution and the amount of K.E. lost by the impact.

3. A truck weighing 10 tons runs at 6 miles per hour into a stationary truck weighing 15 tons. The impact brings the first truck to rest. Find the loss of K.E. in the impact between the trucks.

4. Three balls of masses 30 gms , 20 gms., and 7 gms. respectively lie in a straight line on a smooth horizontal table. The two smaller balls are initially at rest. The first ball impinges directly on the second,

which thereafter strikes the third ball, which moves off with a velocity equal to the initial velocity of the first ball. Find the coefficient of restitution between the balls.

5. The masses of three spheres A, B, C, are $7m$, $7m$, m, their coefficient of restitution is unity, their centres are in a straight line and C lies between A and B. Initially A and B are at rest and C is given a velocity along the line of centres towards A. Show that it strikes A twice and B once and that the final velocities of A, B, C are proportional to 21, 12, 1.

6. Three smooth spheres whose masses are 5, 3 and 4 lbs. are placed in order in a straight line The 5-lb sphere is projected with a velocity of 10 ft./sec. towards the 3-lb. sphere. Show that there will be three impacts and find the subsequent velocities of the spheres if the coefficient of elasticity is ·8.

7. Three small equal spheres are projected simultaneously from the corners of an equilateral triangle with equal velocities towards the centre of the circumscribed circle of the triangle, and meet near the centre. Prove that they return to the corners with velocities diminished in the ratio $e : 1$.

8. Two railway carriages B and C of m lbs. and m' pounds stand on the same line of perfectly smooth rails separated by a short distance; a third carriage A of m lbs. impinges on B and then consequently B impinges on C; prove that A will impinge a second time on B if m' is greater than $\frac{2em}{1+e^2}$, where e is the coefficient of restitution.

[**Solution.** Let u be the velocity with which A impinges on B, and let v and v_1 be the velocities of A and B after the impact. Then, by the law of conservation of momentum,

$$mv+mv_1=mu$$

or
$$v+v_1=u \qquad \ldots(1)$$

By Newton's law

$$v-v_1=-eu \qquad \ldots(2)$$

$$\therefore \quad v_1=\frac{u(1+e)}{2} \text{ and } v=\frac{u(1-e)}{2} \qquad \ldots(3)$$

Now B impinges on C with a velocity v_1; if v_2 and v_3 are the velocities of B and C after the impact, we have,

$$mv_2+m'v_3=mv_1 \qquad \ldots(4)$$

and
$$v_2-v_3=-ev_1 \qquad \ldots(5)$$

Multiplying (5) by m', and adding to (4), we have

$$v_2(m+m')=v_1(m-em')$$

or
$$v_2=\frac{m-em'}{m+m'}\,\frac{u(1+e)}{2} \qquad \ldots(6)$$

Now A and B impinge again, if $v>v_2$

or
$$\frac{u(1-e)}{2}>\frac{m-em'}{m+m'}\,\frac{u(1+e)}{2}$$

or $$(m+m')(1-e) > (m-em')(1+e)$$

or $$m' > \frac{2em}{1+e^2}.\ ]$$

OBLIQUE IMPACT

120. In Art. 113 we distinguished between a direct and an oblique impact. Two bodies are said to impinge obliquely, when the velocity of at least one of them is not along the common normal at the point of contact.

121. Laws of oblique impact. In Art. 114 we enunciated the laws for direct impact. The laws of oblique impact may be stated in a slightly modified form.

1. The law of conservation of momentum. In this case it is important to note that the components of the velocities of the colliding bodies perpendicular to the common normal do not suffer any change after the impact, the reason being that the spheres are perfectly smooth and the forces during impact are wholly along the common normal.

Again, the only force acting on the bodies is the impulse of the blow along the common normal, so that the total momentum in that direction is conserved and is unaltered by impact. This will be explained further when we study the oblique impact of two spheres.

Hence the sum of momenta of the two bodies along the normal remains unaltered by impact between them.

2. Newton's experimental law. When two bodies impinge obliquely their relative velocity resolved along their common normal after impact bears a constant ratio to their relative velocity before impact resolved in the opposite direction. The constant ratio is called the coefficient of restitution.

122. Oblique impact of a perfectly smooth sphere on a perfectly smooth fixed plane. Let AB be a perfectly smooth plane. Let u and v be the velocities of a smooth sphere before and after the impact respectively, making angles α and β with the line CD perpendicular to the plane.

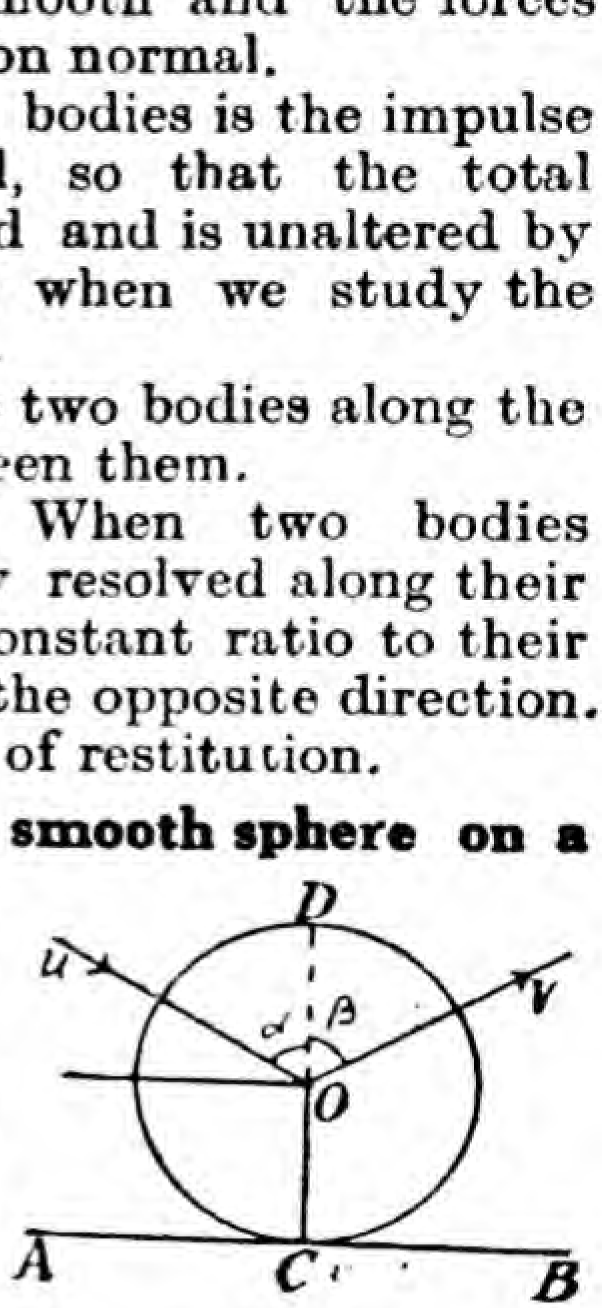


Fig. 60

Colliding bodies being perfectly smooth, there is no force of friction

between the sphere and the plane. Hence the velocity of the sphere along the plane remains unaltered.

$$u \sin \alpha = v \sin \beta \qquad \ldots(1)$$

By Newton's experimental law

$$v \cos \beta = eu \cos \alpha \qquad \ldots(2)$$

Eliminating β between (1) and (2), we get

$$v = u \sqrt{(\sin^2\alpha + e^2 \cos^2\alpha)} \qquad \ldots(3)$$

Dividing (1) by (2)

$$\tan \beta = \frac{1}{e} \tan \alpha \qquad \ldots(4)$$

If we know u and α and e, then using (3) and (4) v and β can be determined.

The impulse of the force of impact on the plane is equal and opposite to the impulse of the force of impact on the sphere, and is therefore measured by the change in the momentum of the sphere perpendicular to the plane.

Hence the impulse of the blow $= mu \cos \alpha - (-mv \cos \beta)$
$= m(1+e)u \cos \alpha$ [using (2)]

Cor. 1. If the impact be direct. $\alpha = 0$

$\therefore$ From (1) $\beta = 0$ and from (2) $v = eu$.
Hence the direction of motion of a sphere, which impinges directly on a smooth plane, is reversed and its velocity is reduced in the ratio $1 : e$.

Cor. 2. If $e = 1$ *i.e.*, the balls are perfectly elastic

$$\alpha = \beta \text{ and } v = u.$$

Hence when the plane is perfectly elastic the angle of reflexion is equal to the angle of incidence, and the velocity is unaltered in magnitude.

Cor. 3. If $e = 0$, then from (2) $\beta = \frac{\pi}{2}$

and $v = u \sin \alpha$.

Hence a sphere after impact with an inelastic plane slides along the plane with its velocity parallel to plane unaltered.

Ex. 1. An imperfectly elastic particle is projected from a point in a horizontal plane with a velocity u at an elevation α. If e be the coefficient of restitution, show that it ceases to rebound from the plane at the end of time $\dfrac{2u \sin \alpha}{g(1-e)}$.

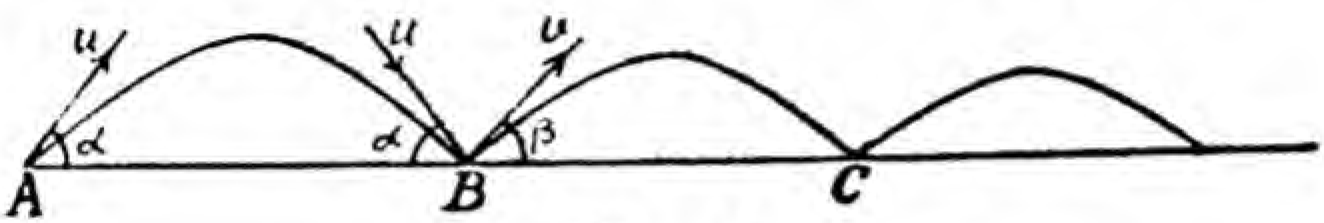


Fig. 61

Let the particle be projected from A. Suppose it strikes the plane at B. The particle describes a parabola in time

$$\frac{2u \sin \alpha}{g} \text{ (By Art. 106)} \qquad \ldots(1)$$

Due to oblique impact at B, we have

$$v \cos \beta = u \cos \alpha \qquad \ldots(2)$$
$$v \sin \beta = eu \sin \alpha \qquad \ldots(3)$$

$$\therefore \qquad v = \frac{eu \sin \alpha}{\sin \beta}.$$

The time for describing the second parabola is

$$= \frac{2v \sin \beta}{g} = \frac{2u \sin \alpha}{g} \cdot e \qquad \ldots(4)$$

Since the particle rebounds from the plane an indefinite number of times, the required time

$$= \frac{2u \sin \alpha}{g}(1 + e + e^2 + e^3 + \ldots \infty)$$

$$= \frac{2u \sin \alpha}{g(1-e)}.$$

Ex. 2. A smooth circular table is surmounted by a smooth rim whose interior surface is vertical. Show that a ball, whose coefficient of restitution is e, projected along the table from a point in the rim in a direction making an angle $\tan^{-1}\sqrt{\dfrac{e^3}{1+e+e^2}}$, with the radius through the point, will

return to the point of projection after two impacts on the rim. Prove that when the ball returns to the point of projection its velocity is to the original velocity as $e^{\frac{3}{2}} : 1$.

Let the ball be projected from A, a point on the rim, making an angle α with the radius OA. It strikes the rim at B and after the impact at B, it moves with a velocity v, making an angle β with OB. It, then, strikes the rim once again at C, and on reflection from C moves straight to the point of projection with a velocity v_1 making an angle γ with OC.

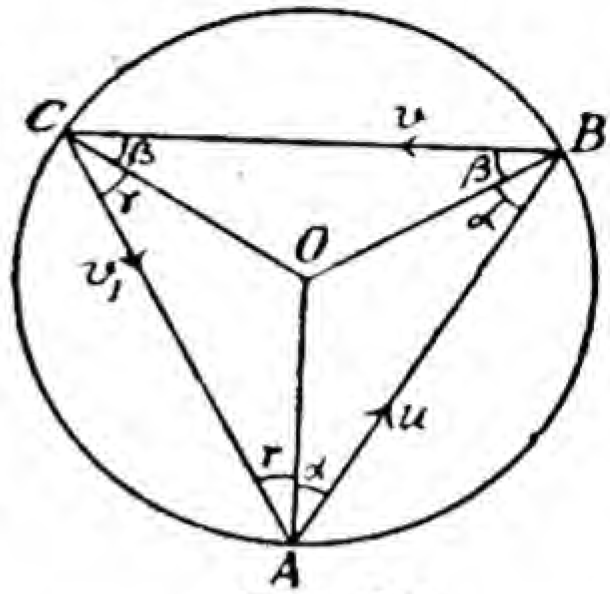


Fig. 62

At B, we have

$$u \sin \alpha = v \sin \beta$$

$$eu \cos \alpha = v \cos \beta$$

$$\therefore \quad \tan \beta = \frac{1}{e} \tan \alpha \qquad \ldots(1)$$

At C, we have

$$v \sin \beta = v_1 \sin \gamma$$

$$ev \cos \beta = v_1 \cos \gamma$$

$$\therefore \quad \tan \gamma = \frac{1}{e} \tan \beta \qquad \ldots(2)$$

Now in the $\triangle$ABC,

$$2(\alpha + \beta + \gamma) = \pi$$

or $$\alpha = \frac{\pi}{2} - (\beta + \gamma)$$

$$\therefore \quad \tan \alpha = \tan\left(\frac{\pi}{2} - \beta - \gamma\right) = \frac{1}{\tan(\beta+\gamma)}$$

$$= \frac{1 - \tan \beta \tan \gamma}{\tan \beta + \tan \gamma}$$

or $$\tan\alpha=\frac{e^3-\tan^2\alpha}{e(e+1)\tan\alpha}$$

or $$\tan\alpha=\sqrt{\frac{e^3}{1+e+e^2}}.$$

Again, $$v_1^2=v^2(\sin^2\beta+e^2\cos^2\beta)$$

and $$u^2=v^2\left(\sin^2\beta+\frac{1}{e^2}\cos^2\beta\right)$$

$$\therefore\quad \frac{v_1^2}{u^2}=\frac{\sin^2\beta+e^2\cos^2\beta}{\sin^2\beta+\frac{1}{e^2}\cos^2\beta}=\frac{\tan^2\beta+e^2}{\tan^2\beta+\frac{1}{e^2}}.$$

$$\therefore\quad \frac{v_1^2}{u^2}=\frac{\frac{\tan^2\alpha}{e^2}+e^2}{\frac{1+\tan^2\alpha}{e^2}}=\frac{\frac{e^3}{1+e+e^2}+e^4}{1+\frac{e^3}{1+e+e^2}}=\frac{e^3(1+e+e^2+e^3)}{(1+e+e^2+e^3)}$$

$$\therefore\quad \frac{v_1^2}{u^2}=e^3$$

or $$\frac{v_1}{u}=e^{\frac{3}{2}}.$$

123. Oblique impact of two perfectly smooth spheres.

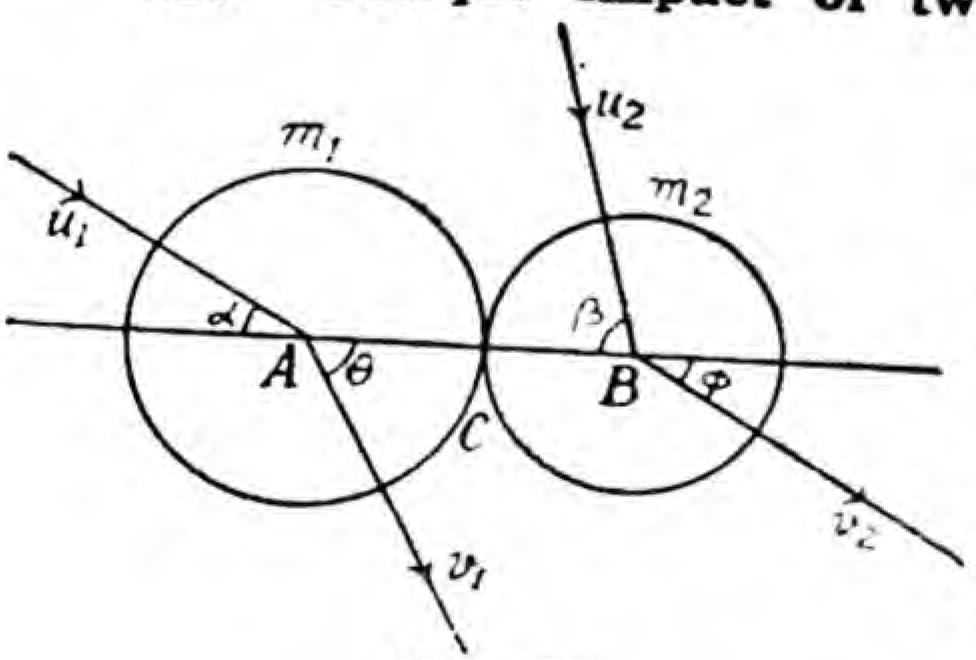


Fig. 63

Let two spheres of masses m_1 and m_2 impinge obliquely. Let AB be their line of centres and also the common normal at C. Let u_1 and u_2 be their velocities before impact making angles α and β with the common normal. Let v_1 and v_2 be their velocities after the impact, making angles θ and ϕ with the common normal.

Given m_1, m_2, u_1, u_2, α and β, to find v_1, v_2, θ, and ϕ.

Since the spheres are perfectly smooth, the components of velocities perpendicular to AB remain unchanged.

$\therefore \qquad u_1 \sin \alpha = v_1 \sin \theta \qquad \ldots(1)$

and $\qquad u_2 \sin \beta = v_2 \sin \phi \qquad \ldots(2)$

Also by Newton's law

$$v_1 \cos \theta - v_2 \cos \phi = -e\,(u_1 \cos \alpha - u_2 \cos \beta) \qquad \ldots(3)$$

Again, by the law of conservation of momentum we have

$$m_1 v_1 \cos \theta + m_2 v_2 \cos \phi = m_1 u_1 \cos \alpha + m_2 u_2 \cos \beta \ldots(4)$$

Multiplying (3) by m_2 and adding the product to (4), we have

$$v_1 \cos \theta = \frac{(m_1 - em_2)\,u_1 \cos \alpha + m_2\,(1+e)\,u_2 \cos \beta}{m_1 + m_2} \qquad \ldots(5)$$

Again multiplying (3) by m_1 and subtracting the product from (4), we get

$$v_2 \cos \phi = \frac{m_1\,(1+e)\,u_1 \cos \alpha - (em_1 - m_2)\,u_2 \cos \beta}{m_1 + m_2} \qquad \ldots(6)$$

By squaring (1) and (5) and adding we get v_1^2 and by division we have $\tan \theta$.

Similarly, by squaring (2) and (6) and adding we get v_2^2 and by division we have $\tan \phi$.

The impulse of the blow on the first ball = the change produced in its momentum $= m_1\,(u_1 \cos \alpha - v_1 \cos \theta)$

$$= \frac{m_1 m_2}{m_1 + m_2}\,(1+e)\,(u_1 \cos \alpha - u_2 \cos \beta).$$

The impulse of the blow on the other ball is equal and opposite to this.

Cor. 1. If $u_2 = 0$, we have from (2) $\phi = 0$ and hence the sphere m_2 moves along the line of centres.

Cor. 2. If $m_1 = m_2$, and $e = 1$, we have

$$v_1 \cos \theta = u_2 \cos \beta, \text{ and } v_2 \cos \phi = u_1 \cos \alpha$$

Hence if two equal perfectly elastic spheres impinge they interchange their velocities in the direction of the line of their centres.

124. Loss of K.E. The expression for the loss of K.E. due to oblique impact of two spheres may be deduced after the manner of Art. 119.

With the same meaning for $m_1, m_2, u_1, u_2, v_1, v_2, \alpha, \theta, \beta, \phi$, as in Art. 123, we get on substituting in (3) of Art. 119,

$v_1 \cos\theta$ for v_1
$v_2 \cos\phi$ for v_2
$u_1 \cos\alpha$ for u_1
and $u_2 \cos\beta$ for u_2, the equation

$$\tfrac{1}{2} m_1 v_1^2 \cos^2\theta + \tfrac{1}{2} m_2 v_2^2 \cos^2\phi = \tfrac{1}{2} m_1 u_1^2 \cos^2\alpha + \tfrac{1}{2} m_2 u_2^2 \cos^2\beta - \frac{1-e^2}{2}\frac{m_1 m_2}{m_1+m_2}(u_1\cos\alpha - u_2\cos\beta)^2 \quad \ldots(1)$$

Also since

$$u_1 \sin\alpha = v_1 \sin\theta \quad \text{and } u_2 \sin\beta = v_2 \sin\phi,$$

we have

$$\tfrac{1}{2} m_1 v_1^2 \sin^2\theta + \tfrac{1}{2} m_2 v_2^2 \sin^2\phi = \tfrac{1}{2} m_1 u_1^2 \sin^2\alpha + \tfrac{1}{2} m_2 u_2^2 \sin^2\beta \quad \ldots(2)$$

Adding (1) and (2) we get

$$\tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 = \tfrac{1}{2} m_1 u_1^2 + \tfrac{1}{2} m_2 u_2^2 - \frac{1-e^2}{2}\frac{m_1 m_2}{m_1+m_2}(u_1\cos\alpha - u_2\cos\beta)^2$$

or The K.E. after impact

$$= \text{K.E. before impact} - \frac{1-e^2}{2}\frac{m_1 m_2}{m_1+m_2} \times (u_1\cos\alpha - u_2\cos\beta)^2$$

Hence we see that in any impact, unless $e=1$, some K.E. is lost. In fact, this energy is transformed into other forms of energy and chiefly appears in the form of heat and sound.

Cor. Suppose, as in the case of a nail hit by a hammer the object struck was at rest.

Putting $u_2=0$, the energy transformed

$$= \tfrac{1}{2}\frac{m_1 m_2}{m_1+m_2}(1-e^2)\, u_1^2 \cos^2\alpha.$$

If the hammer strikes the nail on the head in the direction of the common normal, we get on making $\alpha=0$,

$$\text{Loss of K.E.} = \tfrac{1}{2}\frac{m_1 m_2}{m_1+m_2}(1-e^2)u_1^2$$

$$\therefore \quad \frac{\text{Mechanical energy lost by the blow}}{\text{Mechanical energy before the blow}}$$

$$= \tfrac{1}{2}\frac{m_1 m_2}{m_1+m_2}(1-e^2)u_1^2 \div \tfrac{1}{2} m_1 u_1^2$$

$$= \tfrac{1}{2}\frac{m_2}{m_1+m_2}(1-e^2)$$

This latter expression is made smaller if the ratio of m_1 to m_2 be made bigger ; the bigger the mass of the hammer compared to the mass of the nail, the smaller is the loss of mechanical energy at the impact.

Ex. 1. A smooth elastic ball impings on another at rest. Prove that the balls will move at right angles to one another if the coefficient of elasticity is equal to the ratio of the masses.

Let the ball whose mass is m_2 be at rest initially. Let another ball of mass m_1 strike it with a velocity u_1 making an angle α with the line of centres. Let the velocities of the two balls after the impact be v_1 and v_2 as marked in the figure making angles θ and ϕ with AB, such that

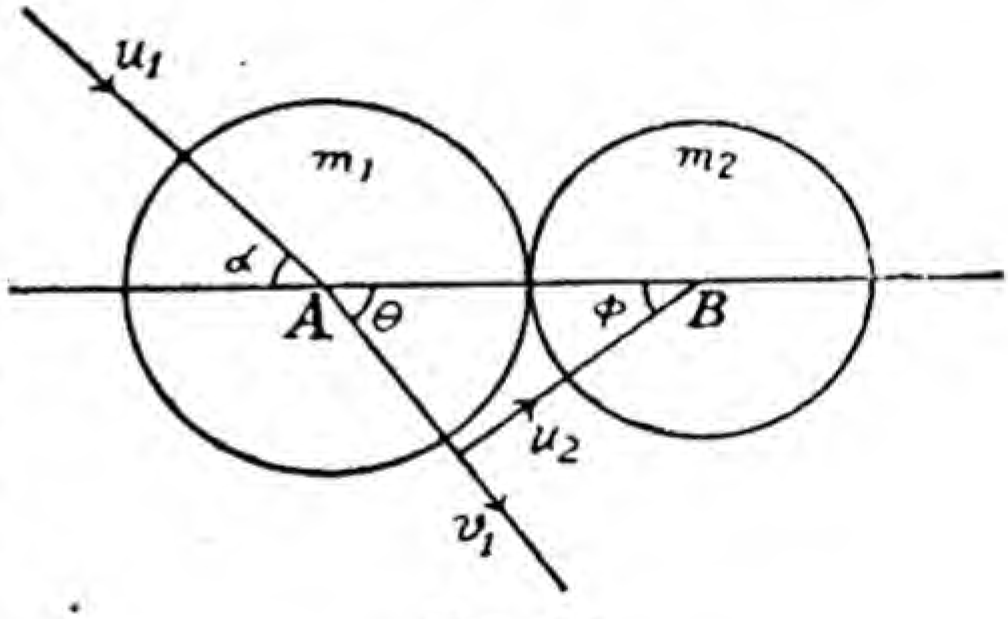


Fig. 64

$$\theta+\phi=\frac{\pi}{2} \qquad \ldots(1)$$

Since the velocities perpendicular to AB remain unaltered, we have

$$v_1 \sin\theta = u_1 \sin\alpha \qquad \ldots(2)$$
$$v_2 \sin\phi = 0 \qquad \ldots(3)$$

$$\therefore \quad \phi = 0 \text{ and } \theta = \frac{\pi}{2}$$

i.e., the second ball of mass m_2 moves along AB, and the incident ball moves perpendicular to AB after the impact. Equation (2) becomes

$$v_1 = u_1 \sin \alpha \qquad \text{...(4)}$$

By the experimental law of Newton

$$v_2 = -e(0 - u_1 \cos \alpha)$$

or

$$v_2 = eu_1 \cos \alpha \qquad \text{...(5)}$$

By the law of conservation of momentum

$$m_1 u_1 \cos \alpha = m_2 v_2 \qquad \text{...(6)}$$

$$\therefore \qquad \frac{m_1}{m_2} = \frac{v_2}{u_1 \cos \alpha} = \frac{eu_1 \cos \alpha}{u_1 \cos} = e.$$

Ex. 2. Two balls whose masses are m and m' impinge and their directions of motion after impact are perpendicular to their directions before impact; if α, α' be the angles which their directions before impact make with the line joining their centres, prove that the coefficient of restitution is

$$\frac{m \sin^2 \alpha' + m' \sin^2 \alpha}{m \cos^2 \alpha' + m' \cos^2 \alpha}.$$

Let u and u' be the velocities before impact making angles α and α' with the line of centres. Let v, v' be the velocities after impact making angles $\frac{\pi}{2} - \alpha$, $\frac{\pi}{2} - \alpha'$ with the line of centres.

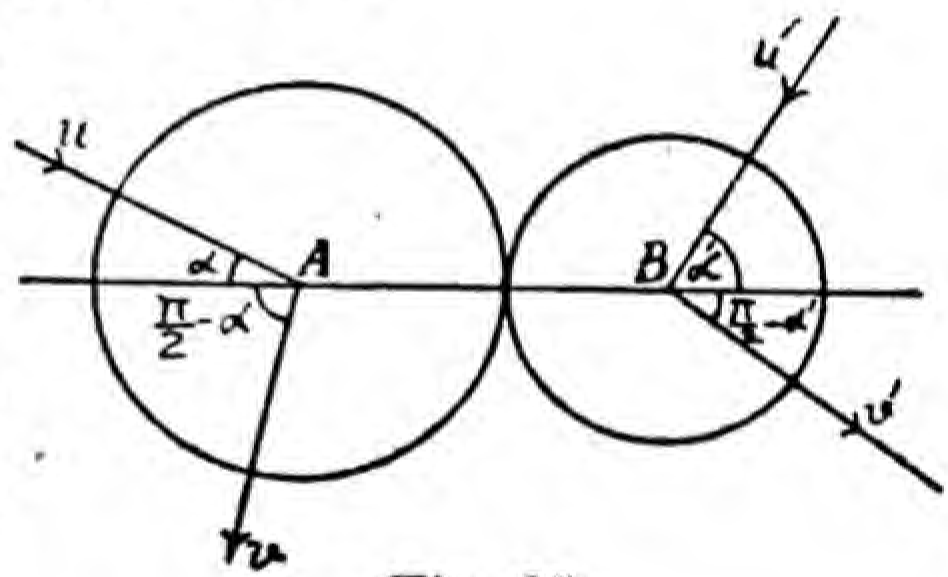


Fig. 65

Then

$$u \sin \alpha = v \cos \alpha \qquad \text{...(1)}$$

$$u' \sin \alpha' = v' \cos \alpha' \qquad \text{...(2)}$$

Also by Newton's law,

$$(-v \cos \alpha - v' \sin \alpha') = -e.[u \cos \alpha - (-u' \cos \alpha')].$$

or

$$v \sin \alpha + v' \sin \alpha' = e(u \cos \alpha + u' \cos \alpha') \qquad \text{...(3)}$$

By the law of conservation of momentum,

$$mu \cos \alpha - m'u' \cos \alpha' = m'v' \sin \alpha' - mv \sin \alpha \qquad \ldots(4)$$

Substituting the values of v' and v from (1) and (2) in (4), we have,

$$mu \left[\cos \alpha + \frac{\sin^2 \alpha}{\cos \alpha}\right] = m'u' \left[\cos \alpha' + \frac{\sin^2 \alpha'}{\cos \alpha'}\right]$$

or

$$\frac{mu}{\cos \alpha} = \frac{m'u'}{\cos \alpha'} \qquad \ldots(5)$$

From (3)

$$e = \frac{v \sin \alpha + v' \sin \alpha'}{u \cos \alpha + u' \cos \alpha'}$$

$$= \frac{\dfrac{u \sin^2 \alpha}{\cos \alpha} + \dfrac{u' \sin^2 \alpha'}{\cos \alpha'}}{u \cos \alpha + u' \cos \alpha'}$$

Replacing u' by $\dfrac{mu \cos \alpha'}{m' \cos \alpha}$ from (5), we get

$$e = \frac{\dfrac{u \sin^2 \alpha}{\cos \alpha} + \dfrac{mu \sin^2 \alpha'}{m' \cos \alpha}}{u \cos \alpha + \dfrac{mu}{m'} \dfrac{\cos^2 \alpha'}{\cos \alpha}}$$

$$= \frac{m' \sin^2 \alpha + m \sin^2 \alpha'}{m' \cos^2 \alpha + m \cos^2 \alpha'}.$$

Examples XV

1. A sphere A of elasticity e impinges with $20\sqrt{2}$ ft./sec. on an equal sphere B at rest, the line of impact making an angle of 45° with the direction of motion of A ; find the velocity of B after the impact.

2. A sphere A impinges on a sphere B of equal mass ; their velocities before impact are at right angles and equally inclined to the line of impact, and are equal in magnitude ; show that when $e = \dfrac{1}{\sqrt{3}}$, their velocities after impact are inclined at an angle 60°.

3. A particle of elasticity e is projected with velocity u at an angle α to the horizon and after striking a fixed vertical wall at a horizontal

distance h ft. returns to the point of prejection ; prove that

$$hg(1+e)=2u^2e. \sin \alpha \cos\alpha.$$

4. A particle of elasticity e is projected from a point half way between two fixed parallel vertical walls $2a$ ft. apart, in a given direction, and after two rebounds comes back to the point of projection ; find the velocity of projection.

[**Hint** :—Let the angle of projection with the horizon be α, then the time of flight from the horizontal motion is $\dfrac{a}{u \cos \alpha}+\dfrac{2a}{eu \cos \alpha}+\dfrac{a}{e^2 u \cos \alpha}$ and from the vertical motion the time is $\dfrac{2u \sin \alpha}{g}$. Therefore $u^2=\dfrac{ag}{\sin 2\alpha}\left[1+\dfrac{2}{e}+\dfrac{1}{e^2}\right]$.

5. A ball impinges on another ball of same mass moving with the same speed in a direction perpendicular to its own, the line joining the centres of the balls at the instant of impact being perpendicular to the direction of motion of the second ball. If e be the coefficient of restitution, show that the direction of motion of the second ball is turned through an angle $\tan^{-1}\left(\dfrac{1+e}{2}\right)$.

6. A perfectly elastic ball is projected on a smooth rectangular billiard table in a direction parallel to one of its diagonals ; find the condition that after impact on each of the four sides the ball will return to the point of projection.

7. A smooth small sphere of elasticity e slides down a smooth inclined plane of height h and inclination α and impinging on a smooth horizontal plane at the foot of the first describes a parabola ; find the rage on the horizontal plane.

8. An elastic ball, of small radius, sliding along a smooth horizontal plane with a velocity of 16 ft./sec. impinges on a smooth horizontal rail at right angles to its direction of motion; if the height of the rail above the plane be one half the radius of the ball, show that the latus rectum of the parabola subsequently described is one foot in length.

9. A ball falling vertically strikes with velocity u a smooth plane inclined at an angle of 30°. If the coefficient of restitution be $\frac{1}{2}$, show that the range of the first bounce is $\dfrac{3u^2}{4g}$.

10. Two elastic spheres, equal in all respects, are moving towards each other with equal velocities, their centres being on two parallel lines whose distance apart is $d_1 (d_1 < d)$ which is the diameter of either sphere. Prove that after impact they will move away from each other with equal velocities, so that their centres are on two parallel lines whose distance apart d_2 is given by the equation

$$d_2^2[e^2 d^2+(1-e^2)d_1^2]=d^2 d_1^2$$

CHAPTER X

MOTION ALONG A CURVED PATH

125. When a particle moves in such a manner that its path obtained by joining its positions at different instants is a curved line, it is said to move along a curved path.

Explanation. In choosing our laws of motion we have accepted the dictum : Nature acts in the simplest way. This simplicity postulate has always been the guiding principle of all researchers in different departments of knowledge. Accordingly, the state of rest or of uniform motion in a straight line is looked upon as an ideal state, in which all the forces acting on the particle are in equilibrium. Any deviation from the ideal state can be explained by introducing the doctrine of unbalanced forces. Thus, in the case of accelerated rectilinear motion, a force is supposed to be acting in the direction of motion producing acceleration in its direction. The motion along a curved path can also be explained on this hypothesis. In this case, the forces are acting in the most general manner and if we keep ourselves confined to coplanar motions, the forces at any instant can be resolved along two perpendicular directions. There are three modes of describing these forces.

(*i*) The forces are resolved along the axes of co-ordinates, namely, OX and OY.

(*ii*) The forces are resolved along the tangent and the normal at any point of the curved path.

(*iii*) They are resolved along OP (the radius vector) and along a line perpendicular to OP at the point P. These forces are the measures of accelerations which a particle may have in these directions, and account for its motion along a curved path.

126. Velocity along a curved path. If P be the position of a particle at any instant of its motion on a curved path, and if s be the length of the arc in the direction of motion in the small time t following the instant under consideration, then the ultimate value of $\frac{s}{t}$, as the time t is taken smaller and smaller, is the measure of the velocity of the moving point at the instant under consideration.

Fig. 66

The velocity at P is entirely in the direction of the tangent at P. The velocity of the particle P in the direction of the normal at P is zero.

127. Angular velocity. The angular velocity of a particle moving in a plane, about a given point in that plane is the rate of change of the angle which the line joining the fixed point to the moving point makes with a fixed straight line in the plane of motion.

If P and Q are two positions of a moving particle, then the ultimate value of $\frac{\theta_1}{t_1}$, [$\angle QOP=\theta_1$, and t_1 is the time of motion from P tv Q], when t_1 is taken smaller and smaller, is the measure of the angular velocity of P about theorigin.

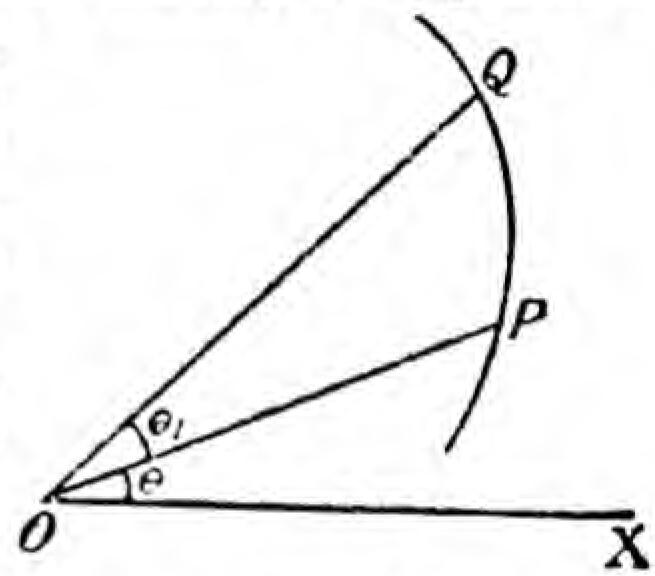


Fig. 67

If the angular velocity be uniform the particle is displaced through equal angles in equal times, and the angle turned through in a given time is called the angular displacement of the particle about the fixed point.

The angular velocity is denoted by the Greek letter 'ω' (omega), and it represents the angle in radians which a particle turns through in one second. The angle θ, described in t seconds, is equal to ωt radians.

128. Particle moving on a circular path. Let O be the centre of the circular path, ω be the angular velocity of the particle about O, and $\angle POA = \theta$, be the angle turned through in time t.

Then

$$\theta = \omega t$$

Also $$\theta = \frac{\text{Arc AP}}{\text{radius OA}} = \frac{vt}{r}$$

[v is the tangential velocity of the particle].

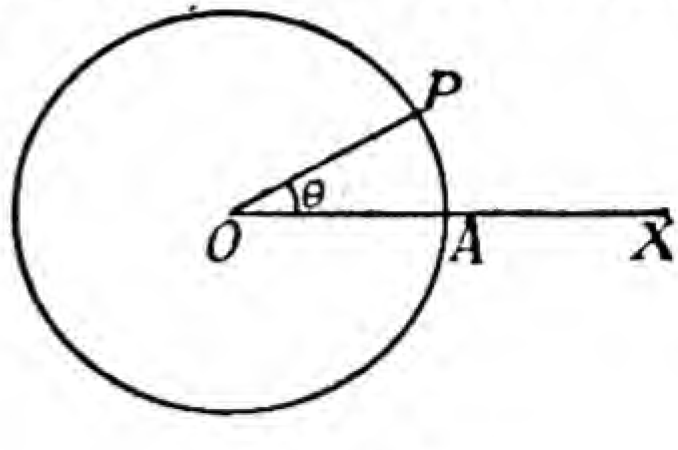


Fig. 68

$$\therefore \quad \omega t = \frac{vt}{r}$$

or $$a = \frac{v}{r}.$$

Roll no 12-S II year
S.P. College

129. Normal and Tangential accelerations. Let A be the position of the particle at any instant, and P its position at the end of an indefinitely small interval of time t.

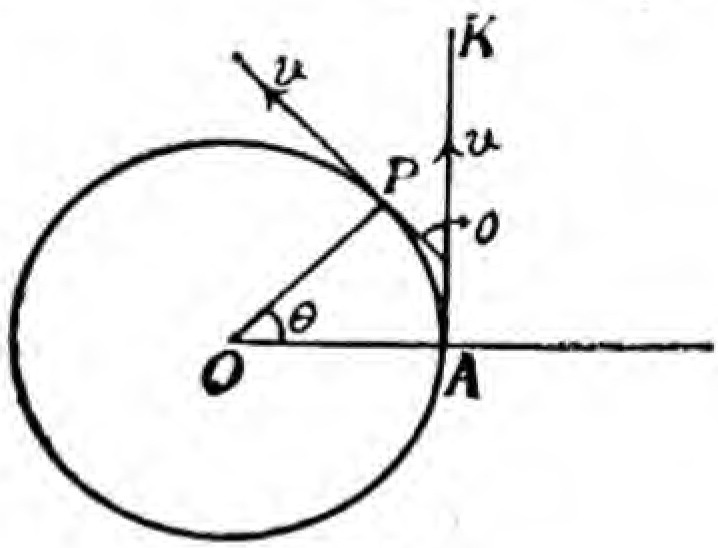


Fig.69

The acceleration along AO = Rate of change of velocity in the direction AO.

$$= \frac{\text{(Resolved part of the velocity at P along AO)} - \text{(Resolved part of the velocity at A along AO)}}{\text{Time from A to P}}$$

$$= \frac{v \sin \theta - 0}{t}$$

$$= \frac{v\theta}{t} \qquad \text{[when } \theta \text{ is small, } \sin \theta = \theta]$$

$$= v\omega$$

$$= \frac{v^2}{r} = r\omega^2.$$

The acceleration along AK = Rate of change of velocity along AK.

$$= \frac{\text{(Resolved part of the velocity at P along AK)} - \text{(Resolved part of velocity at A along AK)}}{\text{Time from A to P}}$$

$$= \frac{v \cos \theta - v}{t} \qquad \text{[when } \theta \text{ is small, } \cos \theta = 1]$$

$$= \frac{v - v}{t}$$

$$= 0.$$

There is no acceleration along the tangent.

Ex. 1. If a particle P is moving along a circle with a constant speed v, find its angular velocity about any point A on the circle.

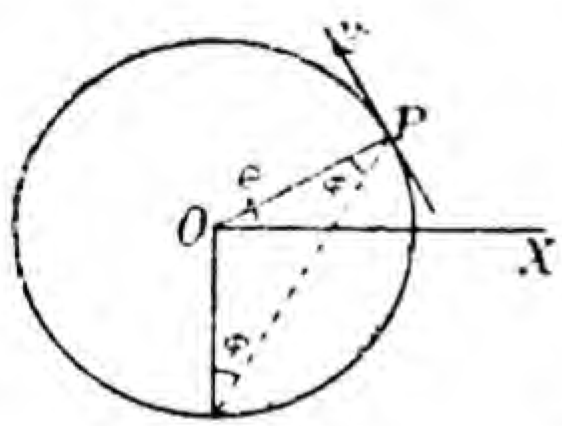


Fig. 70

Let O be the centre of the circular path, which the particle P is describing. If AP makes an angle ϕ with OA, then $AP = 2r \cos \phi$.

The particle P has a velocity v in the direction of the tangent at P. Its component in a direction perpendicular to AP is $v \cos \phi$.

$\therefore$ the angular velocity of P about $A = \dfrac{v \cos \phi}{AP}$

$$= \frac{v \cos \phi}{2r \cos \phi} = \frac{v}{2r}.$$

Ex. 2. A wheel rolls uniformly on the ground, without sliding, its centre describing a straight line; find the velocities of different points of its rim.

Let O be the centre and r the radius of the wheel, and let v be the velocity with which the centre advances. Let A be the point of the wheel in contact with the ground at any instant.

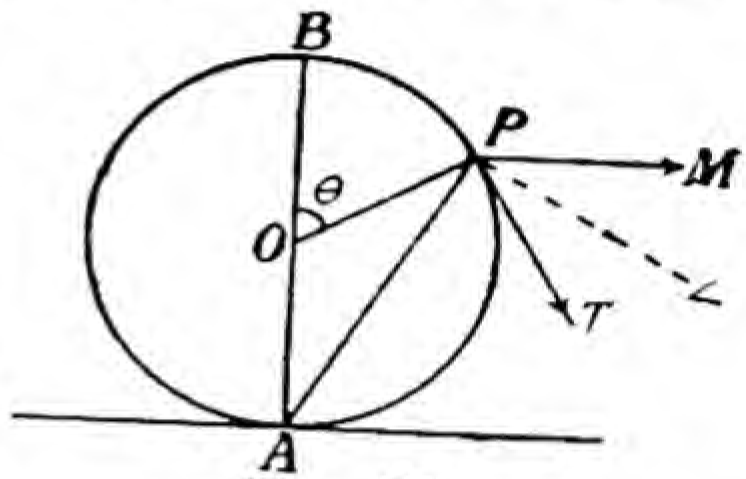


Fig. 71

Now the wheel turns uniformly round its centre whilst the centre moves forward in a straight line; also, since each point of the wheel in succession touches the ground, it follows that any point of the wheel describes the perimeter of the wheel relative to the centre, whilst the centre moves through a distance equal to the perimetor; hence the velocity of any point of the wheel relative to the centre is equal in magnitude to the velocity v of the centre.

Hence any point P of the wheel possesses two velocities each equal to v, one along the tangent, PT, at P to the circle, and the other in the direction, PM, in which the centre O is moving.

Hence the velocity of $A = v - v = 0$, and so A is at rest for the instant.

So the velocity of $B = v + v = 2v$.

Consider the motion of any other point P. It has two velocities, each equal to v, along PM and PT respectively.

Now, since PM and PT are respectively perpendicular to OB and OP, the $\angle MPT = \angle POB = \theta$ (say).

The resultant of these two velocities is a velocity $2v \cos\frac{\theta}{2}$ along PL, where $\angle LPT = \frac{1}{2} \angle MPT = \frac{\theta}{2} = \angle OPA$.

Hence $\angle APL = \angle OPT =$ a right angle.

Hence the direction of motion of the point P is perpendicular to AP, and its angular velocity about A

$$= \frac{2v \cos\frac{\theta}{2}}{AP} = \frac{2v \cos\frac{\theta}{2}}{2r \cos\frac{\theta}{2}} = \frac{v}{r}.$$

= the angular velocity of the wheel about O.

Hence each point of the wheel is turning about the point of contact of the wheel with the ground, with a constant angular velocity whose measure is the velocity of the centre of the wheel divided by the radius of the wheel.

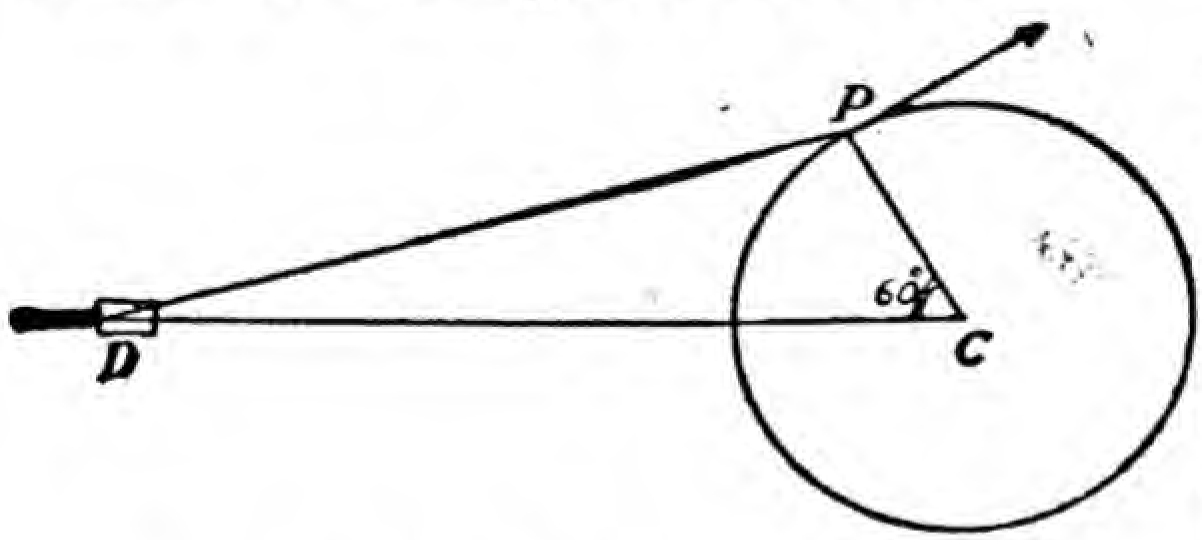
Ex. 3. An engine is travelling at 60 miles an hour, and its wheel is 4 feet in diameter; find the velocity and direction of motion of each of the two points of the wheel which are at a height of 3 feet above the ground.

[*Ans.* $60\sqrt{3}$ m.p.h. at $\pm 30°$ to the horizon.]

Ex. 4. A string has one end attached to the corner of a square board, fixed on a smooth horizontal table, and is wound round the square carrying a particle at its other end; the particle is projected with velocity u at right angles to the side of the square whose length is a; if the length of the string be $4a$, find the time that the string takes to unwrap itself from the square, assuming that the speed of the particle remains the same throughout.

[*Ans.* $\dfrac{5\pi a}{u}$ units of time.]

Ex. 5. The crank of a steam-engine is 9 inches and is rotating at 360 revolutions per minute. The connecting rod is 30 inches long. Find the velocity of the piston when the crank makes an angle 60° with CD.



CP represents the crank, PD the connecting rod, and DC the line of stroke. The cross-head D is fixed to the piston and will therefore have the same motion as the piston.

The velocity of $P = r\omega$

$$= \frac{2\pi \times 360}{60} \times \frac{9}{12}$$

$$= 9\pi \text{ feet per second.}$$

Let us give to D and P a velocity equal and opposite to that of P. This brings P to rest and D's resultant motion will be the relative velocity of D to P, which must be perpendicular to DP. If v is the velocity of D, then the relative velocity of D

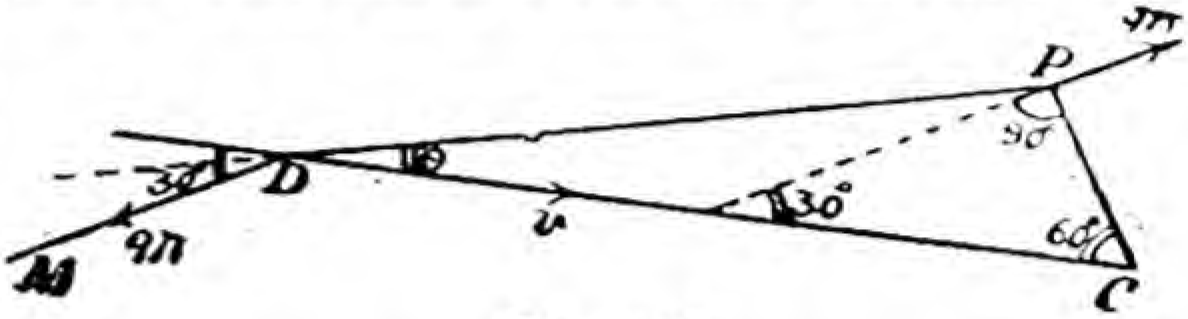


Fig. 73

is resultant of a velocity 9π along DM and v along DC. This resultant is perpendicular to DP. Therefore resolving along DP, we get

$$v \cos \theta = 9\pi \cos (30-\theta)$$

$$v = 9\pi \frac{\cos 30 \cos\theta + \sin 30 \sin\theta}{\cos\theta} = \frac{9\pi}{2} (\sqrt{3} + \tan \theta).$$

But, in the $\triangle$ PDC

$$30 \sin \theta = 9 \sin 60.$$

$$\text{or } \sin \theta = \frac{9\sqrt{3}}{2 \times 30} = \frac{3\sqrt{3}}{20}.$$

$$\therefore \quad \tan \theta = \frac{3\sqrt{3}}{\sqrt{373}}.$$

Hence

$$v = \frac{9\pi}{2} \left(\sqrt{3} + \frac{3\sqrt{3}}{\sqrt{373}} \right) \text{ ft./sec.}$$

130. Centripetal and Centrifugal force. If a body is moving with constant speed in a circular path, the first law tells us that there must be a force acting on it. This force cannot be acting in the direction of motion, otherwise its speed could not remain constant. Therefore, we conclude that the force must be acting at right angles to the direction of motion, that is, along the radius of the circle. This force due to Physical Independence of forces does not produce any effect on the magnitude of the speed of the particle in its circular path.

If the body is moving with speed v ft./sec. in a circular path of radius OA ($=r$ ft.) and if at any moment it is at A and a fraction of second later it is at D, it will be seen that if no force had been acting the body would have travelled along AB from A to C.

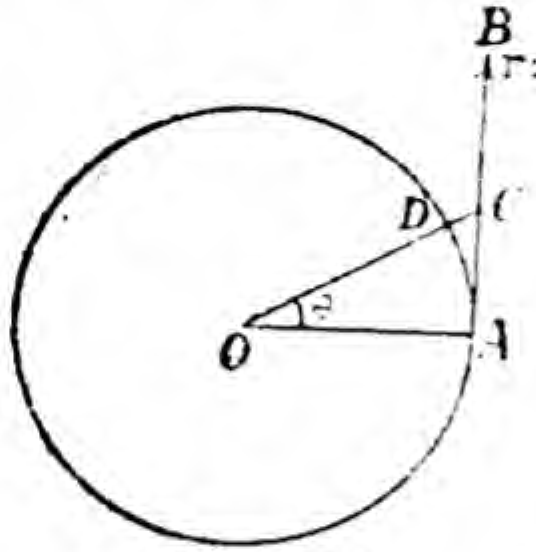


Fig. 74

Therefore, a force must be acting from A to O which would move the body from C to D in the time taken to travel from A to D. In Art. 129 we calculated the normal acceleration $\frac{v^2}{r}$ or rw^2 which could only have been caused by a **centripetal (toward the centre) force** $\frac{mv^2}{r}$ or mrw^2 acting upon the body.

In the case of a particle, which is tied to one end of a light string and whirled in a circle by holding the other end of the string in the hand, the necessary centripetal force is supplied by the tension of the string.

Again, the force may be caused by the pressure of a material curve by means of which the body is constrained to move in a curve ; for example, a train may be made to describe the curved portion of a railway line by means of the pressure of the rails on the flanges of its wheels.

The force may also be of the nature of attraction such as exists between the sun and earth, and which compels the earth to describe a curve about the sun.

Sometimes the force is provided by the internal stress set up in the material of the body. It is well known that by rotating bodies at sufficiently large speeds so great internal stresses are set up that the material cannot withstand and the body flies to pieces. Many cases have occurred where the engine flywheels have suddenly flown to pieces due to governor sticking, and the speed increasing greatly beyond the normal.

From the above examples it is clear that there acts upon the rotating body a force F directed towards the centre of its circular path. The magnitude of this force is $\frac{mv^2}{r}$ or mrw^2.

Hence, we have

$$F = mrw^2 \quad \text{..........} \quad (1)$$

or

$$F - mrw^2 = 0 \quad \text{..........} \quad (2)$$

This second equation admits of an interesting interpretation. It states that the forces F and $(-mrw^2)$ taken together form a system of forces in equilibrium. Hence if we impose upon the rotating body an imaginary force equal in magnitude to mrw^2 and acting away from the centre of rotation, we reduce the dynamical problem to a statical one. This *imaginary* force is called the **centrifugal force**.

It should be understood quite clearly that the particle has no wish or tendency to move in the direction AO. If the particle were to get lose at any moment during the course of motion it would fly off at tangent to the path. The centripetal force simply deflects it from its straight path along AB.

131. The Conical Pendulum. *A mass of m lbs. is suspended from a string of length l and is rotating in a horizontal circle of radius r. Find the time of one revolution and also the tension in the string.*

Such an arrangement is called a conical pendulum.

We will treat this problem firstly as dynamical and secondly as statical.

(1) Dynamical.

Let T be the tension in the string, and ω the angular velocity.

The resultant force vertically $= T\cos\theta - mg$

The resultant force horizontally $= T\sin\theta$

The rate of change of momentum vertically $= 0$.

The rate of change of momentum horizontally (towards C) $= mrw^2$.

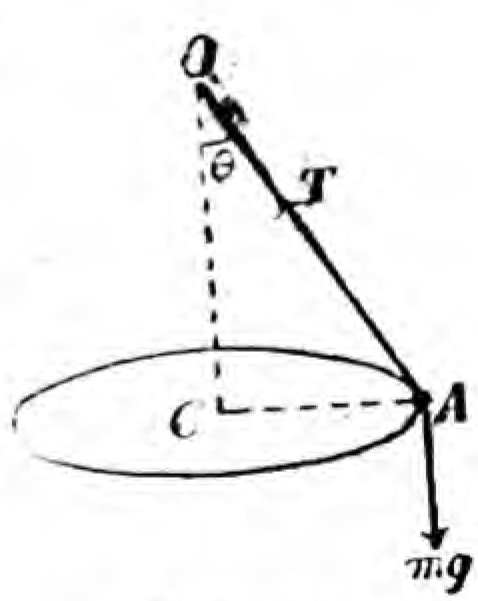


Fig. 75

From the second law of motion,

$$T\cos\theta - mg = 0 \quad \text{........} \quad (1)$$

and

$$T\sin\theta = mrw^2 \quad \text{........} \quad (2)$$

From (1) and (2)

$$\tan\theta = \frac{w^2 r}{g}.$$

i.e. $$w^2 = \frac{g}{r \cot \theta}.$$

$$= \frac{g}{h}, \text{ where } h = OC,$$

$$\therefore \quad w = \sqrt{\frac{g}{h}}.$$

For one revolution the angle turned through $= 2\pi$.

$\therefore$ The time for one revolution $= \dfrac{2\pi}{w}$

$$= 2\pi \sqrt{\frac{h}{g}}.$$

From (1) $T = \dfrac{mg}{\cos\theta} = \dfrac{mgl}{h}$, where $OA = l$.

(2) Statical.

Apply a force mrw^2 as shown in the diagram. We may now treat the mass m as in equilibrium under the action of the three forces, T, mrw^2, and mg.

Resolving vertically and horizontally, we have

$$T \cos \theta - mg = 0$$
$$T \sin \theta - mrw^2 = 0$$

From these we get

$$T = \frac{mgl}{h}$$

and $$w = \sqrt{\frac{g}{h}}$$

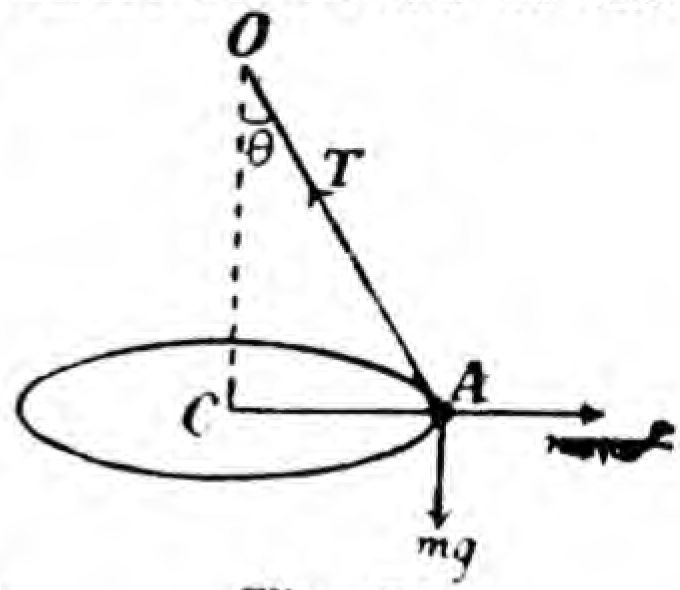


Fig. 76

132. Governor of Steam engines. Old steam-engines were provided with centrifugal governors to control the speed of the engine. They are constructed on the principle of conical pendulum.

In the Watt governor the balls are suspended by links which are pivoted on the axis of rotation and as they rotate they are subjected to a downward force due to their own weight, and the outward centrifugal force. The weight remains constant, but the centrifugal force $\left(\frac{W}{g} r w^2\right)$ increases as the speed increases, so that the balls rise higher the higher the speed. As the balls rise higher they cause the sleeve to rise, which in its turn is connected by a lever mechanism to a valve. The rise in the sleeve partly closes the valve and prevents the speed rising further.

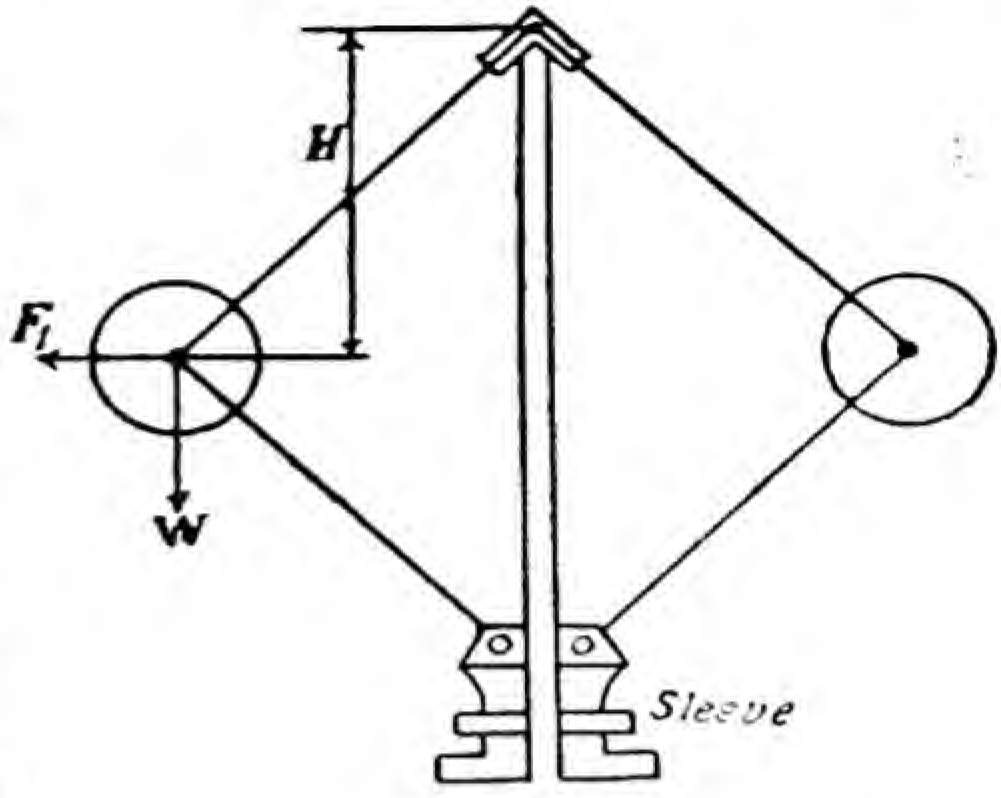


Fig. 77

133. Motion of bicycle rider on a circular path. When a man is riding a bicycle on a curved path he always inclines his body and the machine towards the centre of the path. By this means the total reaction of the ground (the resultant of friction and the normal reaction) becomes inclined to the vertical. The vertical component of R balances his weight and the weight of the machine, while the horizontal component of R supplies the necessary centripetal force.

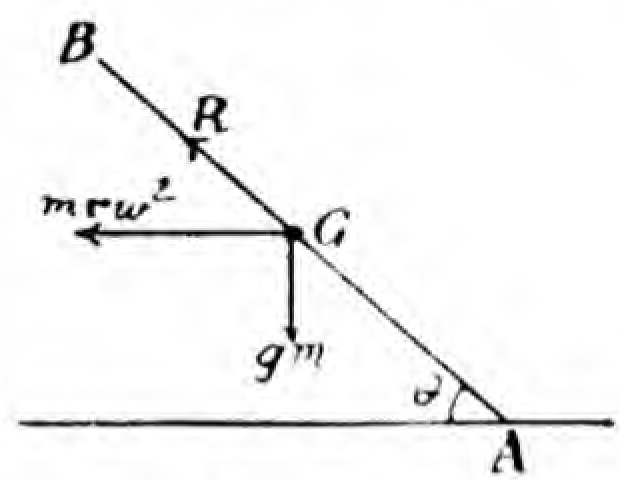


Fig. 78

$$R \cos \theta = m\, r\, w^2 = \frac{m\, v^2}{r} \qquad \text{..........} (1)$$

$$R \sin \theta = m\, g \qquad \text{..........} (2)$$

$$\therefore \quad \tan \theta = \frac{g}{r\, w^2} \qquad \text{..........} (3)$$

134. Motion of a motor car on a circular path. When a car is taking a turn on a perfectly level road the necessary normal acceleration towards the centre of the path is supplied by the total reaction (the resultant of friction and the normal reaction) of the ground. But there is a limit beyond which the force of friction cannot increase and so at high speeds it may be insufficient to supply the necessary amount of centripetal force. There are two possible ways of avoiding overturning of the vehicle at the corner.

Fig. 79

Let us imagine that the speed v will overturn the vehicle if the driver takes it along the path ACB. He can avoid overturning if he takes the car along the path ADB, because the radius of ADB is necessarily larger than the radius of ACB, and he would require less force towards the centre in the second case. It is possible that friction between the road and the tyres may be quite sufficient to supply the centripetal force required in the second case. But this method is not quite safe from the point of view of traffic rules, because the driver may be driving on the wrong side.

Another alternative would be to bank the road by elevating the outer part of the road over the inner part. By so doing the vehicle is tilted a little towards the centre of the path.

In the case of a railway train moving on a level track the centripetal force is supplied by the pressure of the rails on the flanges of the wheels of the carriage. This causes wearing away of the rails, and may be avoided by raising slightly the outer rail above the inner rail, so that the floor of the train is no longer horizontal.

135. Motion of a carriage on a curved level track. Let G be the centre of the carriage, turning round a corner of radius r with a velocity v. Let P and Q be the reactions of the ground, and F_1 and F_2 the lateral thrusts on the flanges of the wheels.

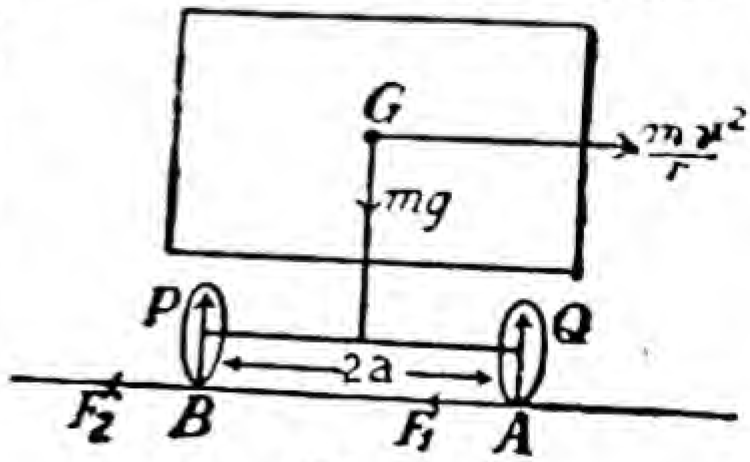


Fig. 80

Impose a centrifugal force $\frac{mv^2}{r}$ and the problem becomes one

of statics. Equating the horizontal and vertical components of forces, we get

$$P+Q = mg \qquad \text{............(1)}$$

$$F_1+F_2 = \frac{mv^2}{r} \qquad \text{............(2)}$$

136. Upsetting of a carriage on a curved level track. The central force $\frac{mv^2}{r}$ necessary to cause the circular motion, should act at G. But in practice it can only act at the points of contact with the rails or ground. This fact is responsible for the upsetting of the carriage rounding a curve at high speed.

In the figure of the last article, taking moments about G, we get

$$Qa-Pa=(F_1+F_2)\,h \qquad \text{where} \qquad AB=2a$$

and h is the height of the C. G. above the horizontal.

Or $$Q-P= \frac{mv^2}{ra}h. \qquad \text{............(3)}$$

Using (1) and (2) and (3), we get

$$Q= \frac{m}{2}\left(g+ \frac{v^2h}{ra}\right)$$

$$P= \frac{m}{2}\left(g- \frac{v^2h}{ra}\right).$$

This shows that Q is always positive and increases with v and h ; while P decreases if v or h increase and vanishes when

$$v^2= \frac{gra}{h} \qquad \text{............(4)}$$

If $v^2 > \frac{gra}{h}$, P becomes negative and the wheels on the inside are no longer in contact with the ground and the carriage begins to tilt.

Hence for safety the C. G. of the carriage should be as near the floor as possible.

137. Cant on railway curves. In railway curves the outer rail is raised above the inner rail by a slight amount,

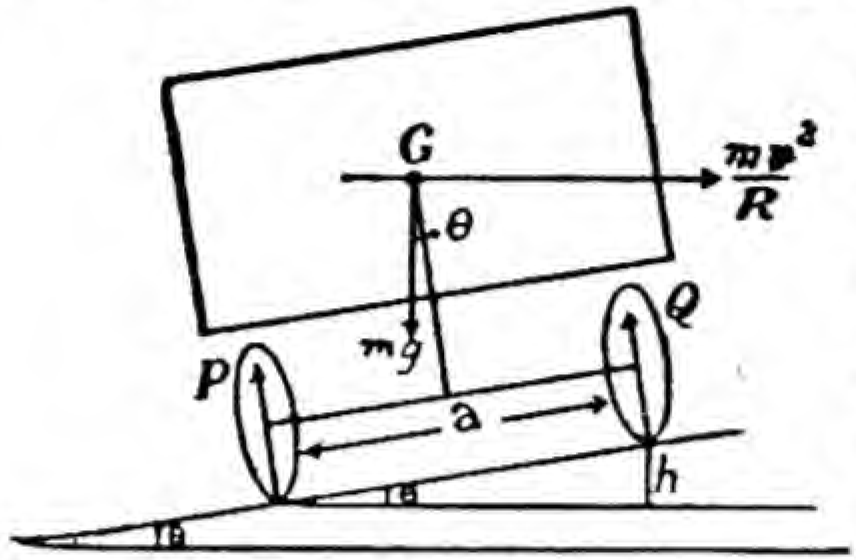


Fig. 81

and by this means for a definite speed, all side thrust on the flanges may be avoided. The amount the outer rail is raised above the inner rail is called the *cant*. If the speed of the train exceeds that for which the cant was calculated the outer flange has to transmit some thrust; if the speed is less than that for which the cant was calculated, the inner flange will bear against the rail.

Let G be the centre of gravity, P and Q the normal thrusts on the wheels, a the mean distance between the wheels, S the flange thrust, and v the speed in feet per second.

The acceleration toward the centre of the curved path described by G is $\frac{v^2}{R}$, where R is the radius of the path.

Impose a centrifugal force $\frac{Mv^2}{R}$, and the problem becomes one of statics.

Resolving all the forces parallel and perpendicular to the track, we have

$$P+Q-Mg\cos\theta-\frac{Mv^2}{R}\sin\theta=0 \qquad \dots (1)$$

$$S+Mg\sin\theta-\frac{Mv^2}{R}\cos\theta=0 \qquad \dots (2)$$

If S is zero, *i.e*, there is no side thrust on the flanges

$$\tan\theta=\frac{v_1^2}{Rg} \qquad \dots (3)$$

where v_1 is the speed for which S is zero.

Cant $=h$

$$=a\sin\theta=\frac{av^2}{\sqrt{v_1^4+R^2g^2}}$$

From (2) and (3)

$$S = \frac{Mv^2}{R}\cos\theta - Mg\sin\theta$$

$$= M\cos\theta\left[\frac{v^2}{R} - g\tan\theta\right]$$

$$= M\cos\theta\left[\frac{v^2 - v_1^2}{R}\right]$$

Hence if v exceed v_1, S is positive and the side thrust is caused by the outer rail. If v be less than v_1, S is negative and the side thrust is caused by the inner rail.

138. Rotating sphere. A hollow smooth sphere is rotating with uniform angular velocity ω about a vertical diameter, to find the position of relative rest of a particle placed inside it.

Let P be the position of the particle

$OP = a$

$\angle PON = \theta$

$\therefore \quad PN = a \sin\theta$

PN is the radius of the circle described by P.

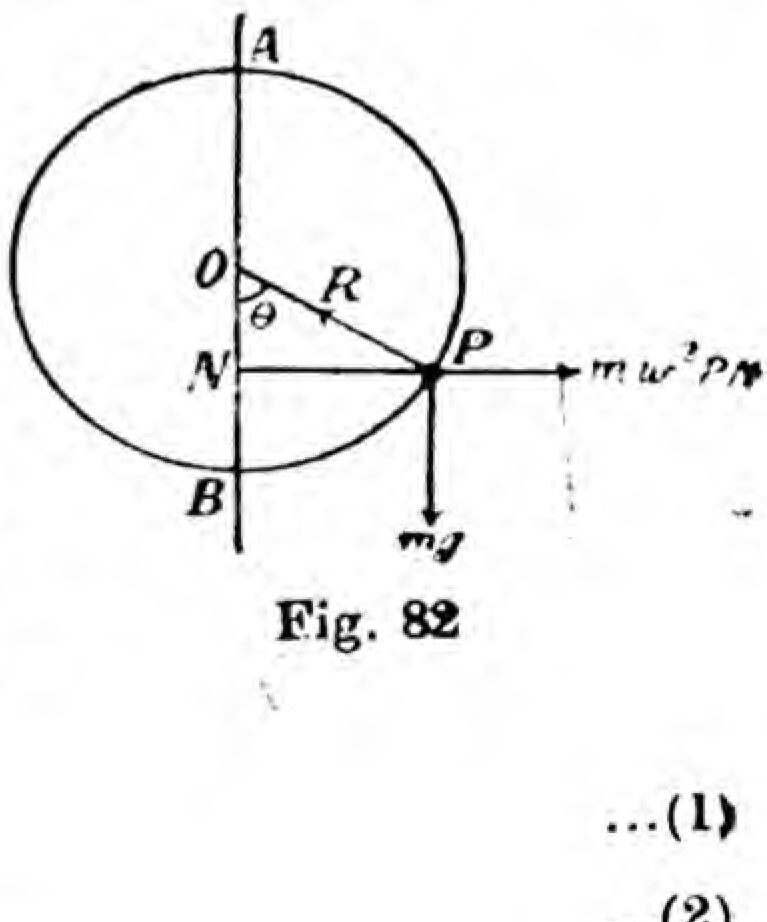


Fig. 82

Let R be the reaction of the sphere on the particle. Impose upon P a centrifugal force $M\omega^2$. PN along NP. The problem becomes one of statics. Hence

$$R\cos\theta = mg \qquad \ldots(1)$$

$$R\sin\theta = m\omega^2 .\ a\sin\theta \qquad \ldots(2)$$

$$R = m\omega^2 a \text{ or } \sin\theta = 0$$

From (1) we get

$$\cos\theta = \frac{g}{aw^2}.$$

If $\sin\theta=0$, then $\theta=0$ and the particle is at rest at B, the lowest point of the sphere.

If $\cos\theta=\frac{g}{a\omega^2}$, the particle is at rest at P.

$$\text{Since } \cos\theta<1$$

$$\frac{g}{a\omega^2}<1$$

$$\therefore \qquad \omega^2>\frac{g}{a}.$$

Hence, if $w<\sqrt{\frac{g}{a}}$, the particle can rest only at the lowest point.

Ex. 1. A particle of mass m, is fastened by a string, of length l, to a point at a distance b above a smooth table; if the particle be made to revolve on the table n times per second, find the reaction of the table. What is the greatest value of n, so that the particle may remain in contact with the table ?

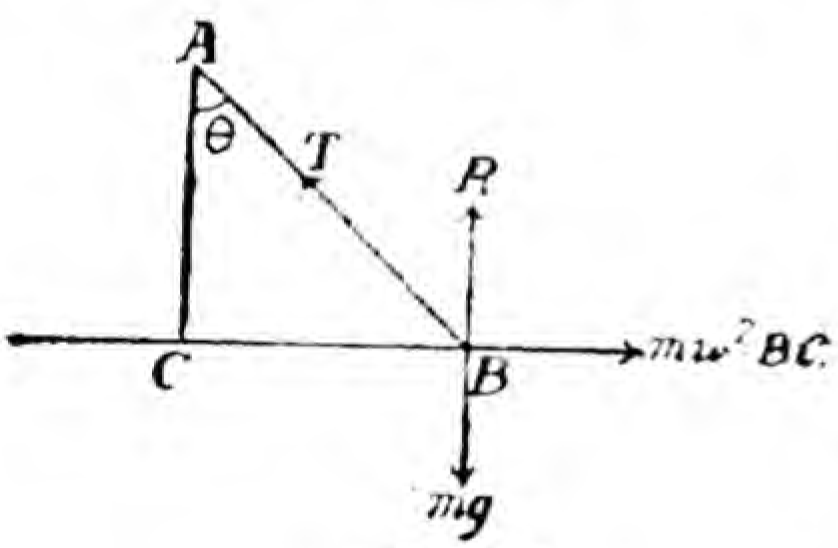


Fig. 83

Let $AC=b$
$AB=l$

Suppose R is the reaction of the table, and T is the tension in the string.

The mass m revolves about C in a circle of radius BC. Impose a centrifugal force $m\omega^2$. BC and the problem becomes statical.

Angular velocity $\omega=2\pi n$

Resolving horizontally and vertically,

$$R+T\cos\theta=mg \qquad \dots (1)$$

and
$$T\sin\theta=m\omega^2.BC$$
$$=4\pi^2n^2m.\,l\sin\theta$$

or
$$T=4\pi^2n^2ml \qquad \dots (2)$$

From (1) and (2)

$$R = mg - 4\pi^2 n^2 ml \cdot \frac{b}{l}$$

$$= m\,(g - 4\pi^2 n^2 b) \text{ poundals} \qquad \ldots (3)$$

The greatest value of n, for which the particle remains on the table is obtained by putting $R=0$ in (3). We get

$$n = \frac{1}{2\pi}\sqrt{\frac{g}{b}}.$$

Ex. 2. A particle, of mass m, on a smooth table is fastened to one end of a fine string which passes through a small hole in the table and supports at its other end a particle of mass $2m$, the particle m being held at a distance c from the hole. Find the velocity with which m must be projected, so that it may describe a circle of radius c.

$AB = c$

Suppose the particle m is made to describe a circular path about A as centre and radius c.

If v is the velocity of the particle, the centripetal force on B is $\frac{mv^2}{c}$. Impose a cen-

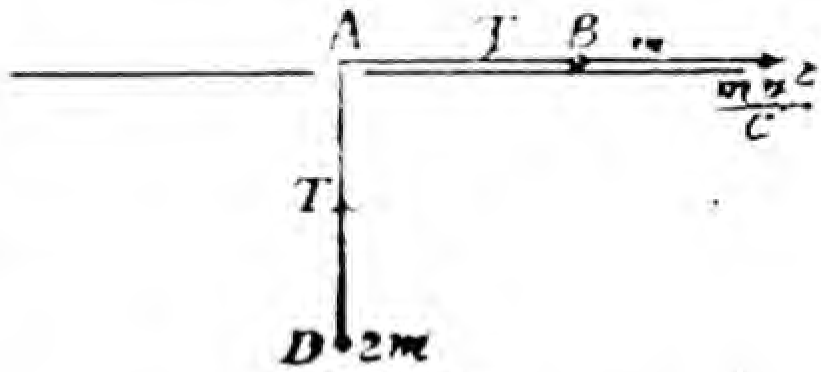


Fig. 84

trifugal force $\frac{mv^2}{c}$ on B, then we have on considering the equilibrium of D and B respectively,

$$T = 2\,mg \qquad \ldots\ldots(1)$$

and

$$T = \frac{mv^2}{c} \qquad \ldots\ldots(2)$$

$$\therefore \qquad \frac{mv^2}{c} = 2mg$$

$$v = \sqrt{2cg}$$

139. Hooke's Law. If an elastic string whose unstretched length is a, is extended by a length l, the tension produced in the string is given by

$$T = \lambda \frac{\text{Extension}}{\text{original length}}$$

$$= \lambda \frac{l}{a}, \text{ where } \lambda \text{ is a constant, called the}$$

modulus of elasticity.

Ex. A body, of mass m, moves on a horizontal table being attached to a fixed point on the table by an extensible string whose modulus of elasticity is λ; given the original length a of the string, find the velocity of the particle when it is describing a circle of radius r.

Let v be the velocity of the particle in its circular path. Hence the centrifugal force $\frac{mv^2}{r}$ must balance the tension of the string.

$$\text{But } T = \lambda \frac{(r-a)}{a}$$

$$\therefore \qquad \lambda \frac{(r-a)}{a} = \frac{mv^2}{r}$$

$$\text{or} \qquad v = \sqrt{\frac{\lambda r(r-a)}{am}}.$$

Examples XVI

1. A string, 5 feet long, can just sustain a weight of 20 lbs; if a mass of 5 lbs. be tied to one end of the string and the string be whirled in a horizontal circle with the other end fixed at the centre of the circle, determine the greatest number of complete revolutions that can be made in one minute by the string without breaking.

2. A bucket of water is swung round in a vertical circle of 26 inches radius. What is the minimum speed of rotation if none of the water is spilled?

3. The spindle EB shown in figure receives vertical support only at E, and is supported horizontally by collars at A and B. DC is a stiff arm rigidly attached to the spindle, and carrying a weight C. If AB is 8 inches, DC 10 inches, D midway between A and B, and the weight at C 10 lbs find the horizontal forces at A and B

Also find these forces when the spindle is rotating freely at 100 revolutions per minute and determine the speed of rotation for which the reaction at A is zero.

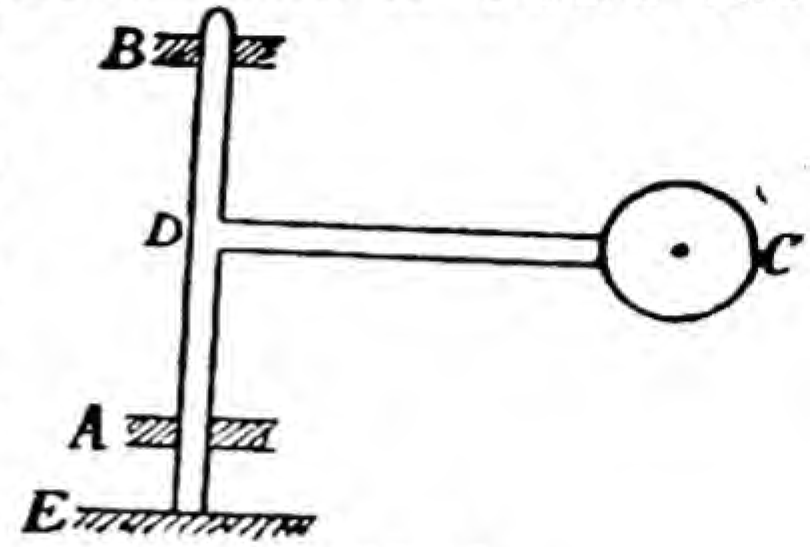


Fig. 85

4 What is the period of a conical pendulum of length 4 feet when the string makes an angle of 30° with the vertical, and what is the tension of the string if the weight at the end is 5 lbs ?

5. A light rod of length 2 feet makes 40 revolutions per minute about the vertical line through the upper end, and carries a mass of $1\frac{1}{2}$ lbs. at the lower end If the rod is free to turn in its own vertical plane find its inclination to the vertical when this inclination is constant and find the tension in the rod.

6. Prove that, if T is the time of revolution of the bob of a conical pendulum at the bottom of a shaft of a deep mine of depth l, the pendulum being suspended from the surface of the earth then the value of g at the bottom of the shaft is given by

$$g=\frac{4\pi^2 l}{T^2}\left(1-\frac{l}{a}\right)$$

where a denotes the radius of the earth.

7. A particle, attached by a light inextensible string to a fixed point, describes a horizontal circle with angular velocity ω, the plane of the circle being at a distance h below the fixed point. Prove that $g=h\omega^2$.

8. If the length of the string is 3 inches and it is inclined at an angle of 60 degrees to the vertical, and the weight of the particle is 1 oz., find the tension in the string and the number of revolutions per minute that the particle is making.

9. A motor car rounds a curve of 150 feet radius on a level road. What is the maximum speed at which this is possible without overturning, when the distance between the wheels is 4 feet and the C. G. of the car is midway between the wheels and 3 feet from the ground ?

10. A cyclist is describing a curve of 60 feet radius at a speed of 10 miles per hour ; find the inclination to the vertical of the plane of the bicycle. What is the least coefficient of friction between the bicycle and the road that the cycle may not side-slip ? (Assume that the rider and the machine are in one plane.)

11. An electric tram turns a corner of mean radius 30 feet at a speed of 10 m. p. h., the road being horizontal. The distance between the rails is 4 feet 6 inches Show that in order that the tram car may not overturn, the centre of gravity must not be more than 10 feet high

12. A railway carriage moves on a circular curve ; find the height to which the outer rail must be raised above the inner so that there may be no lateral thrust on the rails if the radius of the curve be 1320 feet, the breadth between the rails 5 feet, and the carriage has a velocity of 45 miles per hour.

13. What is the side pressure between a train weighing 300 tons and the rails, when the train is going round a curve of 120 yds. raidus at 30 miles per hour, the rails being on the same level ? What should be the cant for no side pressure with a speed of 30 miles per hour if the gauge is 4 feet $8\frac{1}{2}$ inches wide ? With this cant what will be the side pressure it the speed is 45 miles per hour ?

14. If the maximum and minimum speeds of trains on a certain curve of radius r are v and u respectively, show that, if the track is banked up so that the sleepers are inclined at β to the horizontal, such that

$$\tan\beta = \frac{v^2+u^2}{2gr}$$

then the outer lateral thrust on the fastest train is equal to the inner lateral thrust on the slowest train, assuming their weights to be equal.

15. If 2θ be the vertical angle of a smooth hollow cone, whose axis is vertical and vertex downwards, show that the distance from its axis of a body, moving in a circle on its surface and making n revolutions per second is

$$\frac{g\cot\theta}{4\pi^2 n^2}.$$

16. Two particles, of the same mass, are fastened respectively to the middle point and one extremity of a weightless string, and are laid upon a smooth table, the other end being fastened to a point in the table

If the string be pulled tight, and the particles be so projected that they always remain in the same straight line, show that the tensions in the two portions of the string are as 3 : 2.

17 A 2-lb. ball A fastened to the end of a string rests on a smooth horizontal table. The string passes without friction through a hole in the table at B, and has a weight of 10 lbs. hanging from the other end. When the ball is rotating round B at the rate of 2 revolutions per second, find the length of string between A and B.

18. A train of mass M is standing on a railway curve of radius a which is banked up to suit a speed v. Show that there is a lateral thrust on the rails of magnitude

$$\frac{v^2 Mg}{\sqrt{v^4+a^2g^2}}.$$

140. Motion on a smooth curve under the action of gravity. *If a particle slide down the arc of any smooth curve in a vertical plane, and if u be its initial velocity and v its velocity after sliding down through a vertical distance h, to show that $v^2=u^2+2gh$.*

I Method: Let the particle start from A with a volocity u. Let B be the point at a depth h below A on the curve, where the velocity of the particle is v.

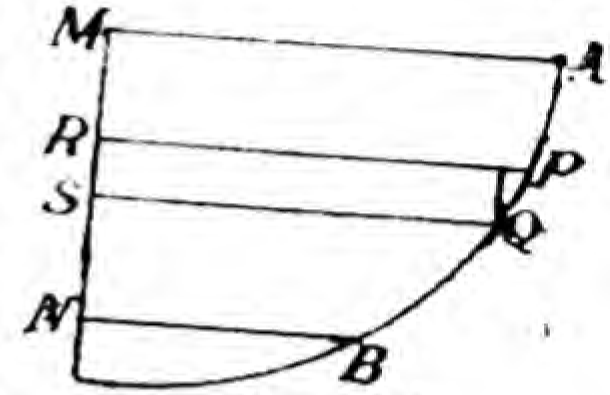


Fig. 86

Let P and Q be two points on the curve, very close to one another. Draw AM, PR, QS and BN perpendiculars on a vertical line, so that $MN=h$. Daw QV vertical to meet PR in V.

The acceleration at P along PQ is g **cos** VQP and if v_p and v_q be the velocities at P and Q, we have

$$v_q^2=v_p^2+2g\cos VQP.PQ$$
$$=v_p^2+2g.\ VQ$$
$$\therefore\ v_q^2-v_p^2=2g.\ VQ$$

i.e., the change in the square of the velocity is due to vertical height between P and Q. Since this is true for every element of arc it is also true for the whole arc AB. Hence the change in the square of velocity in passing from A to B is that due to the vertical height h, so that

$$v^2=u^2+2gh.$$

II Method:. The theorem in the preceding article may be deduced from the Principle of the Conservation of Energy.

The forces acting on the particle are : its weight mg acting vertically downwards and the reaction of the curve acting along the normal inwards. Since the particle moves along the curve no work is done on the body by the normal reaction. The only force that does work is the weight of the body.

Hence the change of kinetic energy in moving from A to B is equal to the work done by the weight of the particle. Therefore

$$\tfrac{1}{2}mv^2-\tfrac{1}{2}mu^2=mgh$$
$$v^2=u^2+2gh.$$

Cor. The velocity at B is independent of the shape of the curve between A and B.

141. If, instead of sliding down the smooth curve, the particle be projected up with a velocity u. so that it moves upwards along a smooth curve, its velocity v when its vertical distance above the point of projection is h, is given by

$$v^2 = u^2 - 2gh.$$

Hence the particle will continue to move up until its velocity becomes zero. By putting $v=0$ in the above formula we see that the particle ascends a height $\dfrac{u^2}{2g}$ above the point of projection.

It will be noticed that the height to which the particle will ascend is independent of the shape of the curve, nor need it continually ascend. The particle may first ascend, then descend, then ascend again and so on. The point at which it will come to rest must be at a height of $\dfrac{u^2}{2g}$ above the point of projection.

Fig. 87

The results of the preceding articles apply to the case of a particle fastened at the end of a light cord, the other end of which is fixed, and the particle is moved in such a manner that the cord is always normal to the path of the particle.

142. Motion on the outside of a vertical circle. *A particle slides from rest at the highest point down the outside of the arc of a smooth vertical circle; to show that it will leave the curve when it has described vertically a distance equal to one-third of the radius.*

Let O be the centre, and A the highest point of the circle. Let v be the velocity of the particle when at a point P of the curve, R the normal reaction of the curve and r the radius of the circle. Draw PN perpendicular to the vertical radius AO, and let $AN = h$.

Since the particle started from rest at A,

$$v^2 = 2g.AN = 2gh.$$

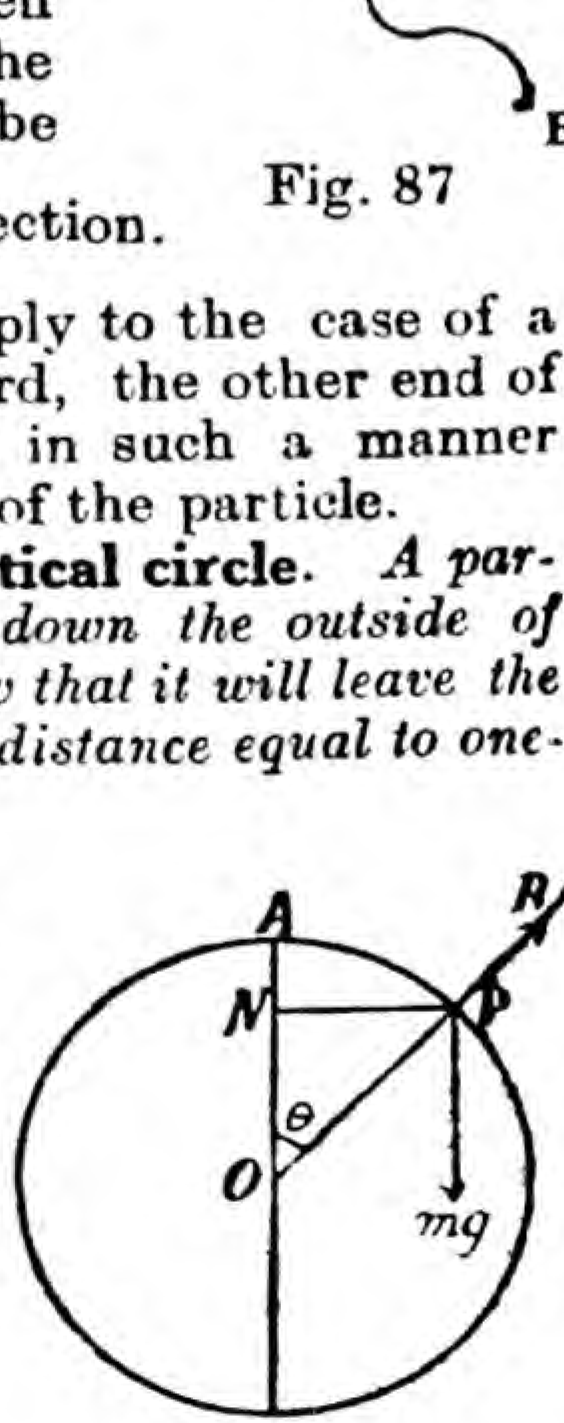


Fig. 88

The force along PO is $mg\cos\theta-\text{R}$, where $\angle\text{POA}=\theta$.

But the force along PO must be $\dfrac{mv^2}{r}$,

$$\therefore\ \frac{mv^2}{r}=mg\cos\theta-\text{R}$$

$$\therefore\ \text{R}=m\left(g\cos\theta-\frac{v^2}{r}\right)$$

$$=m\left(g\,\frac{r-h}{r}-\frac{2gh}{r}\right)$$

$$=\frac{mg}{r}\left(r-3h\right)$$

The particle will leave the curve when R=0. Putting R=0 in the above result we get

$$h=\frac{r}{3}\qquad\therefore\ \text{Cos}\ \theta=\frac{2}{3}$$

The particle will then leave the curve, and describe a parabola freely.

143. Motion in a vertical circle. *A particle of mass m, is suspended by a string, of length r, from a fixed point and hangs vertically. It is then projected with velocity u, so that it describes a vertical circle ; to find the tension of the string and the velocity at any point of the subsequent motion, and to find also the condition that it may just make complete revolutions.*

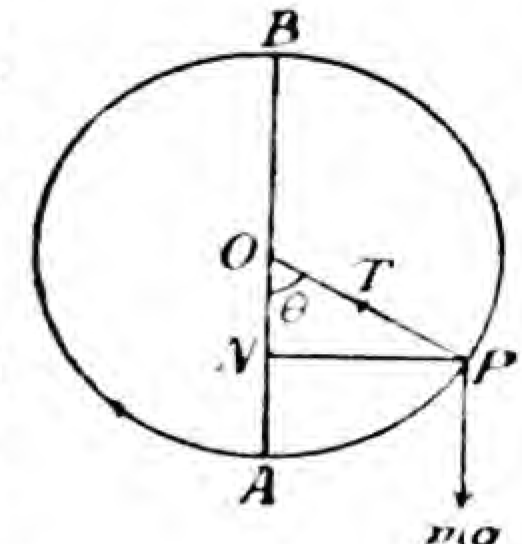


Fig. 89

Let O be the point to which the string is attached and OA the vertical line through O. Let v be the velocity of the particle at any point P of its path, and T the tension of the string there. Let AN be equal to h.

Then,

$$v^2=u^2-2gh \qquad \ldots(1)$$

Since the particle is moving in a circular path, it has a force $=\dfrac{mv^2}{r}$ along the normal PO.

Safdar

$$\therefore \quad \frac{mv^2}{r} = T - mg\cos\theta = T - mg\,\frac{r-h}{r}$$

$$\therefore \quad T = m\,\frac{v^2 + g\,(r-h)}{r} = m\,\frac{u^2 + g\,(r-3h)}{r} \qquad \ldots(2)$$

Equations (1) and (2) determine the velocity and tensions at any point of the path.

The particle will not reach the highest point B if the tension becomes negative ; for then, in order that the particle might continue revolving in a circle, the pull of the string would have to change into a push, and this is impossible in the case of a string.

Hence the particle will just make complete revolutions if the tension vanish at the highest point, where $h = 2r$.

Putting $T=0$ and $h=2r$ in (2), we get,

$$u^2 + g\,(r-6r) = 0$$

or

$$u^2 = 5\,gr.$$

Hence for complete revolutions u must not be less than $\sqrt{5gr}$.

When $u = \sqrt{5gr}$, the tension at the lowest point, by (2)

$$= m\,\frac{5gr + rg}{r} = 6\,mg \text{ poundals.}$$

Hence the string must at least be able to bear a weight equal to six times the weight of the body.

Ex. 1. A particle runs down the outside of a smooth vertical circle, starting from rest at its highest point ; find the latus rectum of the parabola which it describes after leaving the surface.

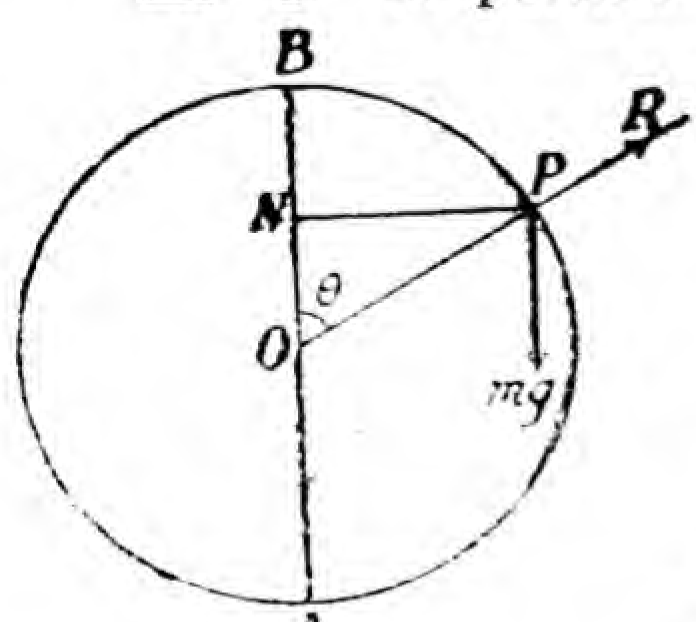


Fig. 90

By Art. 142, the particle leaves the curve when $\cos\theta = \frac{2}{3}$.

At P it is moving with a velocity v given by

$$v^2 = 2gh, \text{ where } BN = h.$$

$$=2g\ (a-a\ \cos\ \theta)=\frac{2ag}{3} \qquad \ldots(1)$$

along the tangent at P, making an angle $(-\dot{\theta})$ with the horizontal.

By Art. 108, the latus rectum $=\dfrac{2v^2\cos^2\theta}{g}$

$$=\frac{2\cdot 2ag\times 4}{3g\times 9}=\frac{16a}{27}.$$

Ex. 2. One of the feats of a trick cyclist is to ride inside a sphere, sometimes riding head downwards past the highest point. If the greatest velocity with which he can reach half way up the sphere is 20 m.p.h., and if he does not pedal in the upper hemisphere, what is the diameter of the largest sphere in which he can perform his feat without losing contact with the sphere ?

Suppose the velocity of the cyclist at B is u. Let OB$=r$ $\angle$POB$=\theta$, and R the normal reaction of the sphere. If v is the velocity at P, the force along PO is $\dfrac{mv^2}{r}$. Hence

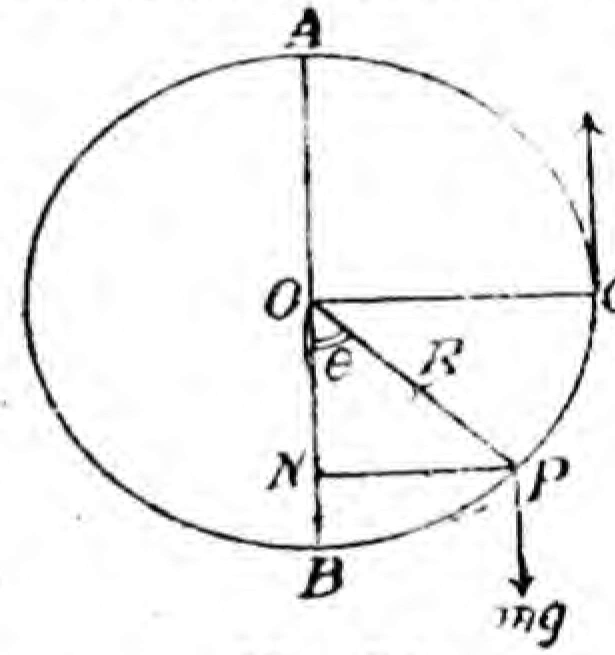


Fig. 91

$$\text{R}-mg\ \cos\ \theta=\frac{mv^2}{r}.$$

But $v^2=u^2-2gh$, where BN$=h$

$$\therefore \quad \text{R}=\frac{m}{r}(u^2-2gh)+mg\frac{(r-h)}{r}$$

$$=m\ \frac{u^2+g\ (r-3h)}{r} \qquad \ldots(1)$$

The cyclist will reach A if R vanishes at A, *i.e.*, when $h=2$ Putting R$=0$ and $h=2r$ in (1) we get,

$$u^2+g\ (r-6r)=0$$

or $\qquad u^2=5gr \qquad \ldots(2)$

$\therefore \quad$ velocity at C$=\sqrt{u^2-2gr}$.

But we are given that at C his velocity is $\frac{88}{3}$ ft./sec.

$$\therefore \quad (\tfrac{88}{3})^2=u^2-2gr$$

$$=5gr-2gr; \qquad [\text{using } (2)]$$

$$\therefore \qquad r=\frac{88\times 88}{27\times 32}$$

$$\therefore \qquad 2r=17\cdot 9 \text{ feet approximately}$$

Examples XVII

1. A particle hangs from a point O by a string of length a. It is projected horizontally with velocity u such that $u^2=(2+\sqrt{3})\,ag$. Show that the string becomes slack when it has described an angle $\cos^{-1}\left(-\frac{1}{\sqrt{3}}\right)$.

2. A particle at the end of a string of length l, the upper end of which is fixed, is projected horizontally with the velocity $\sqrt{ngl}$. If the string becomes slack before the particle reaches the top of the circle, show that it does so at a height $\frac{l}{3}(1+n)$ above the lowest point.

3. A heavy particle is attached to a fixed point by a fine string of length a; the particle is projected horizontally from the lowest point vith velocity $\sqrt{ag\left(2+\frac{3\sqrt{3}}{2}\right)}$. Prove that the string will first become slack, when inclined to the upward vertical at an angle of $30°$, will become tight again when horizontal, and slack again when inclined to the upward vertical at an angle $\cos^{-1}\left(\frac{\sqrt{3}}{8}\right)$.

4. A heavy particle, hanging by an inextensible string of length a from a fixed point, is projected horizontally with a velocity $\sqrt{2gh}$. If $\frac{5}{2}a > h > a$, prove that the circular motion ceases when the particle has reached the height $\frac{1}{3}(a+2h)$. Prove further that the greatest height ever reached by the particle above the point of projection is

$$\frac{(4a-h)(a+2h)^2}{27a^2}.$$

CHAPTER XI

SIMPLE HARMONIC MOTION

144. Theorem. *A particle P describes a circle with uniform angular velocity, and if* Q *be always the foot of the perpendicular drawn from P upon a fixed diameter AOA′ of the circle, to show that the acceleration of* Q *is always directed towards the centre* O *of the circle and varies as the distance of* Q *from* O, *and to find the velocity of P and its time of describing any space.*

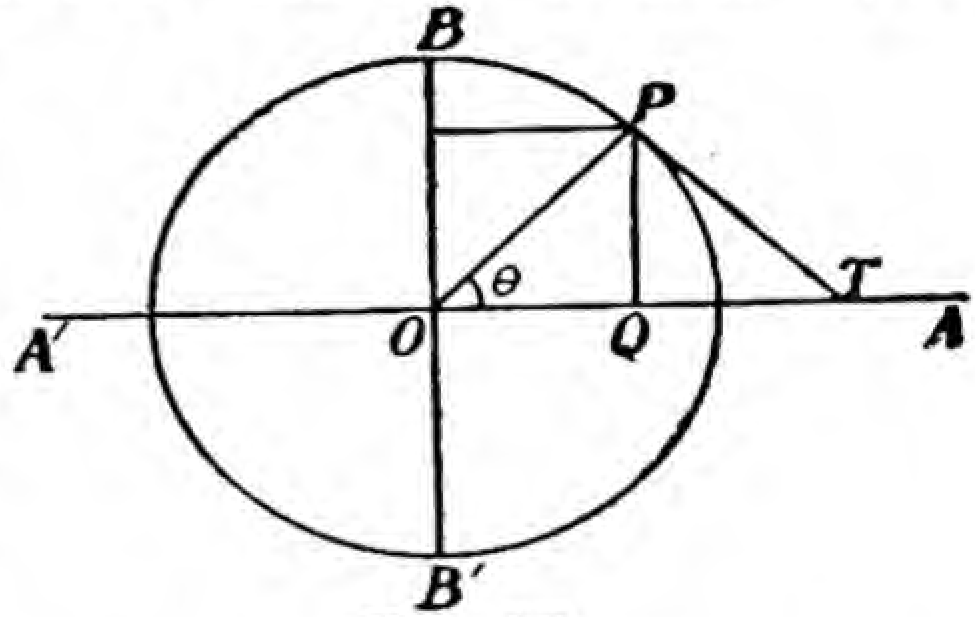


Fig. 92

Let a be the radius of the circle, whose centre is O. Let PT be the tangent at P meeting OA produced at T. Let ω be the constant angular velocity of P. If the particle P started from A, the angle turned through in time t would be $\theta = \omega t$.

Since Q is always at the foot of the perpendicular to AA′ drawn from P, its velocity and acceleration are the same as the resolved parts, parallel to AO, of the velocity and acceleration of P.

But the acceleration of P is $a\omega^2$ directed towards PO.

Hence the acceleration of Q along QO$=a\omega^2 . \cos\theta = \omega^2 .$ OQ and therefore varies as the distance of Q from O.

Also the velocity of Q$=a\omega . \cos$ PTQ$=a\omega . \sin\theta$.

$$=\omega\sqrt{a^2 - x^2} \qquad \ldots\ldots\ldots\ldots \quad (1)$$

where OQ$=x$.

This velocity is greatest at O and is zero at A and A′. Also the acceleration vanishes, and changes its sign, as the point Q passes through O.

The point Q therefore moves from rest at A, has its greatest velocity at O, comes to rest again at A′, and then retraces its path to A. As P moves from A along the circle and completes one revolution, the point Q moves from A to A′ and then back to A, thus making one complete oscillation.

The time in which Q describes any distance AQ
= time in which P describes the arc AP

$$=\frac{\theta}{\omega}=\frac{1}{\omega}\cos^{-1}\frac{x}{a}. \qquad \ldots\ldots\ldots\ldots \quad (2)$$

The time from A to A′ and back again to A = The time taken by P to make one complete revolution round the circle.

$$\therefore \quad T=\frac{2\pi}{\omega} \qquad \ldots\ldots\ldots\ldots \quad (3)$$

145. Simple Harmonic Motion. Definition. *If a point move in a straight line so that its acceleration is always directed towards, and varies from a fixed point in the straight line, the point is said to move with simple harmonic motion.*

The point Q in the last article moves with simple harmonic motion (S. H. M.). If the acceleration of the point be μ.OQ directed towards O, we get, by comparing with the expression for acceleration of Q in the last article, that $\mu=\omega^2$.

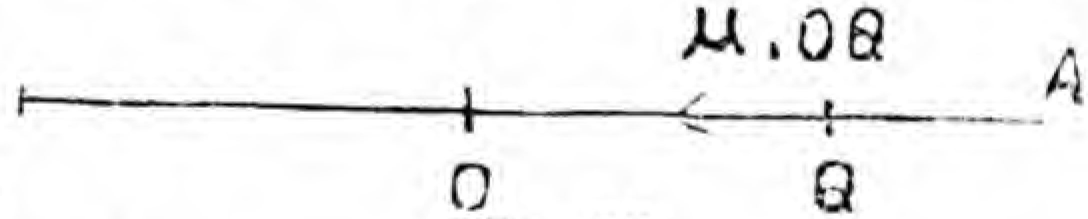


Fig. 93

Hence from (1), (2) and (3) of the last article we get

(1) the velocity of the particle when at a distance x from O is $\sqrt{\mu(a^2-x^2)}$;

(2) the time that has elapsed when the point is at a distance x from O is $\frac{1}{\sqrt{\mu}}\cos^{-1}\frac{x}{a}$ and

(3) the time that elapses before it is again in its initial position is $\frac{2\pi}{\sqrt{\mu}}$.

The point O is called the centre of oscillation. The maximum distance OA or OA′ of the moving point on either side of teh centre O is called the **Amplitude** of motion.

The time that elapses from any instant till the instant in which the moving point is again moving through the same position with the same velocity and direction is called the **Periodic Time** of the motion.

The periodic time $\dfrac{2\pi}{\sqrt{\mu}}$ *is independent of the amplitude of the motion.*

Alternative expression for velocity and distance.

From (2) Art. 144, we have,

$$\frac{\theta}{\omega}=\frac{1}{\omega}\cos^{-1}\frac{x}{a}.$$

But $\theta=\omega t=\sqrt{\mu}\,t$

$$x=a\cos\sqrt{\mu}\,t.$$

And velocity $=\sqrt{\mu\,(a^2-x^2)}$.

$$=\sqrt{\mu}\,a\sin\sqrt{\mu}\,t.$$

Ex. 1. A body is attached to one end of an inextensible string and the other end moves in a vertical line with n complete oscillations per second. Show that the string will not remain tight during the motion unless $n^2<\dfrac{g}{4\pi^2a}$, where a is the amplitude of motion.

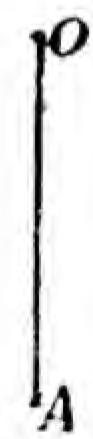


Fig. 93

The end O executes S. H. M. The string can be slack only when O is moving upwards. Its maximum acceleration vertically upwards is $\mu\,a$. (Art. 144).

If the string remains tight during the motion, the acceleration of the particle due to gravity must exceed $\mu\,a$, *i.e.*,

$$g>\mu\,a \qquad \text{...........(1)}$$

But $\quad T=\dfrac{1}{n}=\dfrac{2\pi}{\sqrt{\mu}}$.

or $\qquad \mu=4\pi^2n^2 \qquad \text{...........(2)}$

From (1) and (2) we get,

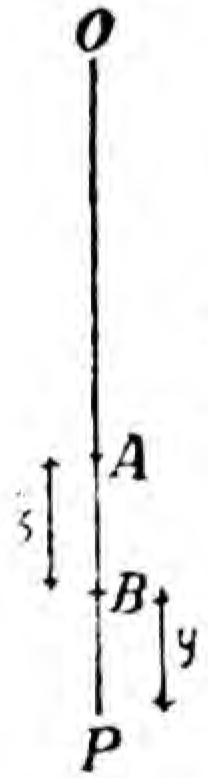
$$n^2<\frac{g}{4\pi^2a}.$$

Ex. 2. A mass of 4 lbs. is suspended from the end of a light helical spring, and when the system is at rest the spring is found to be extended 3 inches. The mass is now slowly depressed a further distance of 2 inches and then let go.

Find (*a*) the periodic time of vibration, (*b*) the maximum velocity.

If T = pull of the spring and x the extension, $T = \frac{\lambda x}{l}$ where λ may be called 'the spring constant', and l is the unstretched length of the spring.

When a mass of 4 lbs. hangs, the spring extends $\frac{1}{4}$ feet.



$$\therefore \quad 4g = \frac{\lambda}{4l}$$

$$\text{or} \quad \frac{\lambda}{l} = 16g \qquad \ldots(1)$$

If at any instant y denotes the displacement of the particle below the equilibrium position, the upward force on the particle is

$$T - 4g = \lambda . \frac{(\frac{1}{4} + y)}{l} - 4g$$

$$= 16g(\tfrac{1}{4} + y) - 4g$$

$$= 16g\, y.$$

Fig. 94

Hence the acceleration of the particle towards the centre of force $= \frac{16gy}{4} = 4gy$

$$\therefore \quad \text{Periodic time} = \frac{2\pi}{\sqrt{4g}} = \frac{\pi}{\sqrt{2}} \text{ seconds.}$$

The velocity is given by

$$v^2 = 4g\left[(\tfrac{1}{6})^2 - y^2 \right]$$

v is maximum, when $y = 0$

$$v_{max} = \sqrt{4g . \tfrac{1}{36}} = \frac{4\sqrt{2}}{3} \text{ ft./sec.}$$

Ex. 3. A light elastic string of natural length l has one extremity fixed at a point A and the other end attached to a stone the weight of which, in equilibrium, would extend the string to a length l_1. Show that if the stone be dropped from rest at A, it will come to an instantaneous rest at a depth $\sqrt{l_1^2 - l^2}$ below the equilibrium position.

Let $AB = l$

When the particle is suspended, the string extends to O such that $AO = l_1$.

When the particle is dropped from A, the string remains unstretched up to B, and its velocity at $B = \sqrt{2gl}$. ...(1)

Suppose at any instant of its subsequent motion, the particle is at P, such that $OP = x$. Then the force on the particle

$= T \quad mg$, where T is the tension of the string and m the mass of the particle.

Fig. 95

Now $$T = \lambda \frac{l_1 - l + x}{l},$$ where λ is the modulus of elasticity.

Since the particle extends the string to a length l_1, we have

$$mg = \lambda \frac{l_1 - l}{l} \quad ......(2)$$

$$\therefore \quad T - mg = \lambda \frac{l_1 - l + x}{l} - \lambda \frac{l_1 - l}{l}$$

$$= \frac{\lambda x}{l} \quad ......(3)$$

Hence the acceleration of the particle $= \frac{\lambda}{ml} x$

Therefore the velocity of the particle is given by

$$v^2 = \frac{\lambda}{ml}(a^2 - x^2) \quad ......(4)$$

where a is the amplitude of S.H.M.

Now when $x=-(l_1-l)$, $v=\sqrt{2gl}$.

$$\therefore \qquad 2gl=\frac{\lambda}{ml}\left[a^2-(l_1-l)^2\right]$$

or
$$2l(l_1-l)=a^2-(l_1-l)^2\,; \qquad \text{[using (2)]}$$
$$a=\sqrt{l_1^{\,2}-l^2}.$$

Ex. 4. One end of an elastic string whose modulus of elasticity is λ and whose natural length is a is tied to a fixed point on a smooth horizontal table, and the other end tied to a mass m lying on the table. The particle is pulled to a distance where the extension of the string is b, and then let go; describe the character of motion and show that the period of one complete oscillation is

$$2\left(\pi+\frac{2a}{b}\right)\sqrt{\frac{am}{\lambda}}.$$

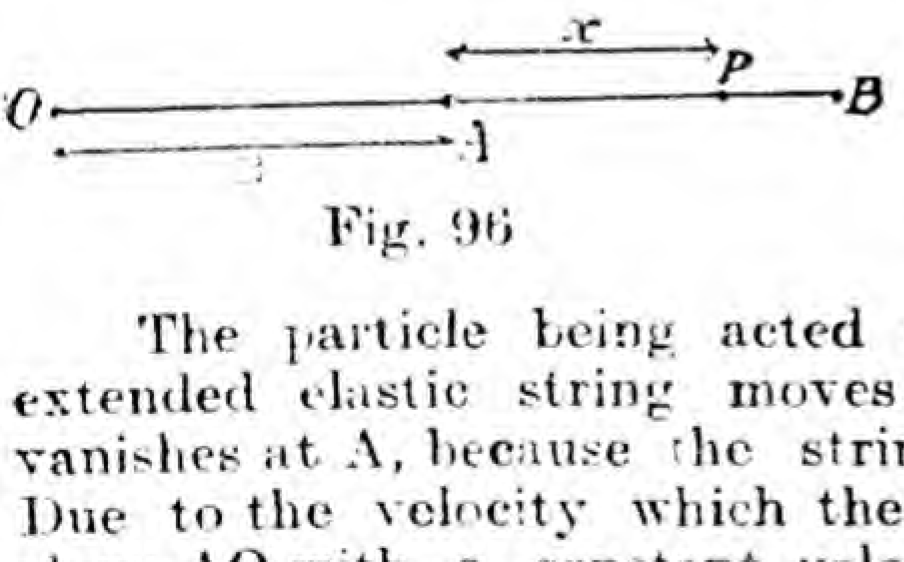


Fig. 96

Let O be the fixed point, and $OA=a$. Let the string be extended to B, where $AB=b$, and released.

The particle being acted upon by the tension of the extended elastic string moves towards A. The tension vanishes at A, because the string regains its natural length. Due to the velocity which the particle has acquired it moves along AO with a constant velocity till it reaches a point A′ on the left of O, such that $OA=OA'$. After this the string begins to extend in the direction of motion till the particle reaches a point B′, such that $OB=OB'$. When the particle has reached B′ its velocity becomes zero and it retraces its path to B.

When the particle is at P, where $AP=x$, its acceleration towards $BA=\dfrac{\lambda}{m}\cdot\dfrac{x}{a}$.

Hence the time for describing the distance $BA=\dfrac{\pi}{2\sqrt{\dfrac{\lambda}{ma}}}$

$$=\frac{\pi}{2}\sqrt{\frac{\lambda}{ma}} \qquad \ldots\ldots(1)$$

The velocity is given by

$$v^2=\frac{\lambda}{am}(b^2-x^2).$$

Hence the velocity at $A=b\sqrt{\frac{\lambda}{am}}$.

$\therefore$ Time for describing the distance $AO=\frac{a}{b}\sqrt{\frac{am}{\lambda}} \qquad \ldots\ldots(2)$

Hence the time from B to $O=\sqrt{\frac{am}{\lambda}}\left(\frac{a}{b}+\frac{\pi}{2}\right)$

$\therefore$ The time for one complete oscillation$=4\sqrt{\frac{am}{\lambda}}\left(\frac{a}{b}+\frac{\pi}{2}\right)$

$$=2\left(\pi+\frac{2a}{b}\right)\sqrt{\frac{am}{\lambda}}.$$

Examples XVIII

1. A point is describing simple harmonic motion making three complete oscillations per second. The extent of motion on either side of the mean position is 2 inches. Calculate the maximum velocity and maximum acceleration in this motion.

Find also the velocity and acceleration when the point is distant one inch from the centre.

2. A spring balance is such that 1 lb. placed on the scale pan, which weighs 1 lb. depresses the pan 0·5 inch. A mass of 0·5 lb. is allowed to fall from rest on to the pan from a height of 2 feet. Neglecting the weight of the spring, show that the maximum distance the pan will be depressed is 2·87 inches nearly.

3. A light spiral spring whose natural length is l cms. and whose modulus of elasticity is the weight of n grammes, is suspended by one end and has a mass of m grammes attached at the other; show that the time of a vertical oscillation of the mass is

$$2\pi\sqrt{\frac{m}{n}\frac{l}{g}}$$

4. A light elastic string stretches $\frac{1}{4}$ inch for every pound of tension. If, when the string is vertical and unstretched, the upper end being

fixed, a weight of 3 lbs is attached to the lower end and then suddenly released, find how far it will fall before coming to rest, and find the time of a complete oscillation.

5. A horizontal shelf moves vertically with S.H.M. whose complete period is one second; find the greatest amplitude in centimeters it can have, so that objects resting on the shelf may always remain in contact.

6. An elastic string, to the middle point of which a particle is attached, is stretched to twice its natural length and placed on a smooth horizontal table, and its ends are then fixed. The particle is then displaced in the direction of the string; find the period of oscillation

[Let l be the length of the elastic string, tied between two fixed points A and B, distant $2l$ apart. Let O be the middle point.

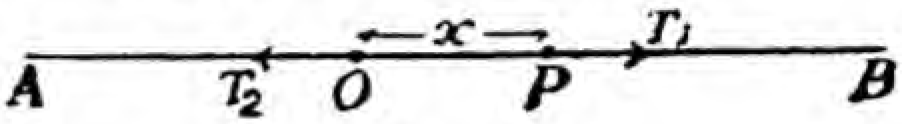


Fig. 97

Let P be the position of the particle when it is displaced towards B, such that $OP = x$.

The extended length of the portion AP of the string $= l + x$

$\therefore$ Extension $= l + x - \frac{l}{2} = \frac{l}{2} + x.$

Similarly, the extension of the portion $BP = (l - x) - \frac{l}{2} = \frac{l}{2} - x.$

If λ is the modulus of elasticity, the tension in

$$PA,\ T_2 = \lambda \cdot \frac{\frac{l}{2} + x}{\frac{l}{2}}$$

and the tension in PB, $T_1 = \lambda \cdot \dfrac{\frac{l}{2} - x}{\frac{l}{2}}$

$\therefore$ The force urging the particle towards the centre O

$$= T_2 - T_1$$

$$= \frac{2\lambda}{l} \cdot 2x = \frac{4x\lambda}{l}.$$

The acceleration of the particle towards $O = \frac{4\lambda}{lm} \cdot x$

$\therefore$ By Art. (3) $T = \dfrac{2\pi}{\sqrt{\frac{4\lambda}{lm}}} = \pi \sqrt{\dfrac{lm}{\lambda}}$].

7 A point P moves in a straight line with S.H.M., the centre of motion being O, and the extreme position being A. A point Q is taken in OA, such that $2OQ^2=OA^2$. Show that the time from A to Q is the same as that from Q to O.

8. A horizontal board is made to perform simple harmonic oscillations horizontally, moving to and fro through a distance of 30 inches and making 15 complete oscillations per minute. Show that the least value of the coefficient of friction is $\frac{5\pi^2}{16g}$ in order that a heavy body placed on the board may not slip.

9. A horizontal board is made to perform simple harmonic oscillations vertically, moving to and fro through a distance of 30 inches and making 15 complete oscillations per minute. Show that a book weighing one pound will not leave the board. Find the greatest and the least pressures exerted by the book.

EXTENSION TO MOTION IN A CURVE

146. Suppose that a moving point P is describing a portion AOA′, of a curve of any shape, starting from rest at A and moving so that its tangential acceleration is always along the arc towards O and equal to μ. arc OP, then the propositions of Art. 145, are true with slight modification.

Let B′O′B be a straight line equal in length to the arc AOA′, and let the arc OP be equal to O′P′. Let P′ be moving towards O′ with an acceleration $\mu . O'P'$.

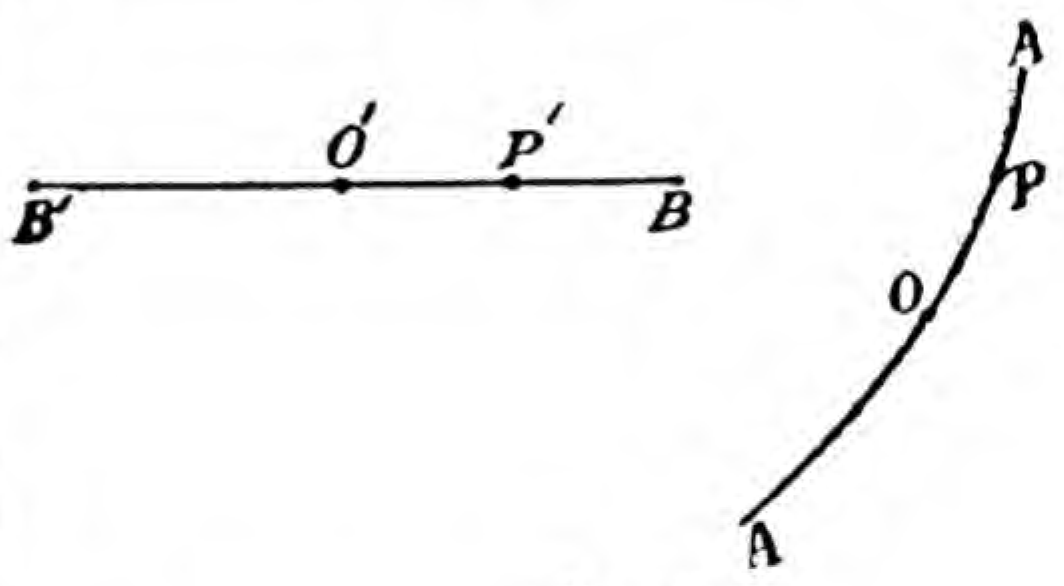


Fig. 98

Since the acceleration of P′ in its path is the same as that of P, the velocities acquired in travelling equal distances are same.

Hence

(1) Velocity of P = velocity of P^1.

$$\sqrt{\mu (O'B^2 - O'P'^2)} = \sqrt{\mu \left\{ (\text{Arc } OA)^2 - (\text{Arc } OP)^2 \right\}}$$

(2) The time from A to P=time from B to P′

$$=\frac{1}{\sqrt{\mu}}\cos^{-1}\left(\frac{O'P'}{O'B}\right)=\frac{1}{\sqrt{\mu}}\cos^{-1}\left(\frac{\text{arc OP}}{\text{arc OA}}\right)$$

and (3) the time from A to A′ and back again $=\dfrac{2\pi}{\sqrt{\mu}}$.

PENDULUMS

147. A simple pendulum is a mass, which may be considered *small*, attached to one end of a light inextensible string, the other end of which is fixed, and which oscillates in a vertical plane.

The time of oscillation depends on the angle through which the string swings on each side of the vertical.

If the angle of oscillation be small, it can be shown that the time of oscillation of the pendulum is approximately constant.

148. Theorem. *If a particle of mass m be tied by a light inextensible string to a fixed point, and allowed to oscillate through a* **small** *angle about the vertical position, to show that the time of a complete oscillation is* $2\pi\sqrt{\dfrac{l}{g}}$, *where l is the length of the string.*

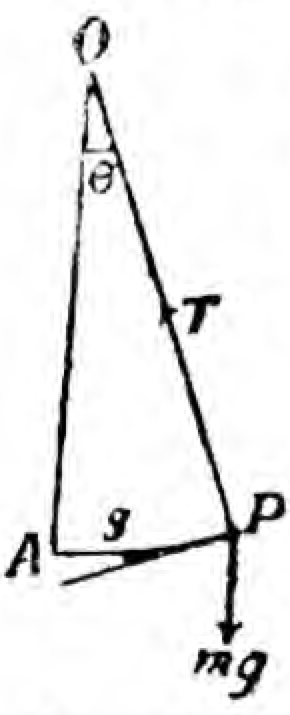


Fig. 99

Let O be the fixed point, OA the vertical line through O, AP a portion of the circular arc described by the particle, and let the angle AOP$=\theta$.

Resolving mg along OP and perpendicular to it, we see, that the component along OT balances T, so that,

$$T=mg\cos\theta \qquad \text{......(1)}$$

The component along PS urges the bob to move along the circular arc AP ; its magnitude is $mg\sin\theta$.

Hence the acceleration of the bob along the tangent

$$=\frac{m.g\sin\theta}{m}=g\sin\theta$$

$$=g\theta, \text{ approximately, if } \theta \text{ is small}$$

$$=\frac{g}{l}\times \text{arc AP} \qquad [\because \text{ arc AP}=l\theta.]$$

The acceleration along the tangent to the path therefore varies as the arcual distance from the centre of oscillation.

By Art. 146 (3), the time of complete oscillation

$$=\frac{2\pi}{\sqrt{\frac{g}{l}}}=2\pi\sqrt{\frac{l}{g}}$$

Note. The first discovery of this principle of the time of swinging of a pendulum is attributed to Galileo about the year 1582. He observed that the great bronze lamp which hangs from the roof of the cathedral at Pisa seemed to have a uniform time of swing, whatever be the arc through which it moved, and he verified the fact by counting the beats of his pulse.

Ex. To find the length of a simple pendulum which makes one complete oscillation in 2 seconds.

$$\text{Periodic time} = 2\pi\sqrt{\frac{l}{g}}$$

$$2= 2\pi\sqrt{\frac{l}{32{\cdot}2}}$$

$$l= \frac{32{\cdot}2}{\pi^2}=3{\cdot}26 \text{ feet.}$$

It may be noted that if we keep the length of the pendulum exactly constant, then for different positions on the earth's surface the periodic time will vary inversely as the square root of g. This gives us a simple method of comparing g at different places.

149. Seconds Pendulum. A seconds pendulum is one which makes one beat (half oscillation) in one second. It takes 2 seconds to complete one oscillation. *The number of beats made during one day is 24×3600; and the number of complete oscillations is 1800×24.*

150. Change in g due to position. Considering the earth to be a sphere it can be shown, that the earth attracts any particle outside itself just as if the whole mass of the earth were concentrated at its centre. Therefore, by Newton's law of gravitation the force of attraction on any particle at a distance r from the centre of the earth and lying outside its surface is $\gamma \frac{mM}{r^2}$, where m is the mass of the particle, M is the mass of earth and γ is the gravitation constant.

Hence the acceleration of the particle $= \frac{\gamma M}{r^2}$

$$\therefore \qquad g = \frac{\mu}{r^2}$$

where μ is a constant.

If we move to a place whose distance from the centr e o the earth is $(r+h)$,the acceleration due to gravity on a particle is g_1.

$$\therefore \qquad g_1 = \frac{\mu}{(r+h)^2}$$

$$\therefore \qquad g_1 = g\left(\frac{r}{r+h}\right)^2$$

$$= g\left(1+\frac{h}{r}\right)^{-2}$$

$$= g\left(1-\frac{2h}{r}+\dots\dots\right), \quad \text{[Binomial expansion]}$$

Hence the acceleration diminishes as we move away from the surface of the earth.

Again for a particle inside the surface of the sphere and at a distance r from the centre, the attraction varies directly as the distance of the particle from the centre.

Hence $\quad g = \lambda r$

If the particle moves a further distance d towards the centre of the earth, so that its distance from the centre is $(r-d)$,

Then,

$$g_2 = \lambda(r-d)$$

$\therefore$

$$g_2 = g\frac{r-d}{r}$$

$$= g\left(1-\frac{d}{r}\right).$$

In this case also the acceleration is reduced.

Hence the acceleration of a particle in the earth's field is maximum when the particle lies on its surface. The acceleration diminishes when the particle is moved both away from the centre or towards it.

151. Loss or gain of oscillations. The time of oscillation of a pendulum depends upon the length of the string l, and on the value of g at the place. A change in l or g or both causes a change in the periodic time, and the pendulum loses or gains a number of vibrations in a certain interval.

If n is the number of vibrations in a certain interval of time T seconds, and t, the time of one vibration or beat

$$T = nt$$

But time for one vibration $= \pi\sqrt{\dfrac{l}{g}}$

$\therefore$

$$\frac{T}{n} = \pi\sqrt{\frac{l}{g}} \qquad \ldots\ldots (1)$$

or

$$n = \frac{T}{\pi}\sqrt{\frac{g}{l}} \qquad \ldots\ldots (2)$$

If g changes to g_1 and l to l_1, such that n_1 is the number of beats in T seconds, then

$$n_1 = \frac{T}{\pi}\sqrt{\frac{g_1}{l_1}} \qquad \ldots\ldots (3)$$

From (2) and (3) we get,

$$\frac{n}{n_1} = \sqrt{\frac{g}{l}} \cdot \sqrt{\frac{l_1}{g_1}} \qquad \ldots\ldots (4)$$

The formula (4) helps us to find the number of beats under changed conditions.

Ex. 1. Find how many seconds a clock would lose per day if the length of its pendulum were increased in the ratio 900 : 901.

In this case g does not change, and $n=24\times60\times60$ seconds.

If $n_1=n+N$, where N is the number of beats gained, we get, by Art. 151,

$$\frac{n+N}{n}=\sqrt{\frac{l}{l_1}}$$

or

$$N=24\times60\times60\left[\sqrt{\frac{900}{901}}-1\right]$$

$$=24\times60\times60\left\{\left[1+\frac{1}{900}\right]^{-\frac{1}{2}}-1\right\}$$

$$=-\frac{24\times60\times60}{2\times900}=-48 \text{ seconds.}$$

Hence the clock loses 48 seconds per day.

Ex. 2. A clock with a seconds pendulum loses 20 seconds per day at a place where the acceleration due to gravity is 32 ft./sec.² Find what change (*i*) in length, (*ii*) in gravity is necessary to make it accurate.

In (4) of Art. 151,

put

$$n=24\times60\times60$$
$$n_1=24\times60\times60-20$$

and

$$l=l_1+L.$$

The gravity remaining same, we have

$$\frac{24\times60\times60-20}{24\times60\times60}=\left(1+\frac{L}{l_1}\right)^{\frac{1}{2}}=1+\tfrac{1}{2}\frac{L}{l_1} \quad \text{(nearly)}$$

$$\therefore \quad \frac{L}{l_1}=2\left\{1-\frac{20}{24\times60\times60}-1\right\}$$

$$\therefore \quad L=-\frac{1}{2160}l_1.$$

Hence the pendulum must be shortened by $\frac{1}{2160}$ of its original length.

(2) In this case, the formula becomes

$$\frac{n}{n_1} = \sqrt{\frac{g}{g_1}}, \text{ because length has not changed.}$$

where
$$n_1 = 24 \times 60 \times 60 - 20$$
$$n = 24 \times 60 \times 60$$
$$g_1 = 32$$

Let
$$g = 32 + G$$

$$\therefore \quad \frac{24 \times 60 \times 60 - 20}{24 \times 60 \times 60} = \left[\frac{32}{32 + G}\right]^{\frac{1}{2}}$$

$$1 - \frac{20}{24 \times 60 \times 60} = \left[1 + \frac{G}{32}\right]^{-\frac{1}{2}}$$

$$= 1 - \frac{1}{2} \cdot \frac{G}{32} \quad \text{(approximately)}$$

$$\therefore \quad G = \frac{64 \times 20}{24 \times 60 \times 60}$$

$$= \cdot 0148 \text{ ft./sec.}^2$$

Thus the acceleration due to gravity must be increased by ·0148 ft./sec.^{2}

Examples XIX

1. A pendulum clock beats seconds. If the clock keeps correct time in one place, how many seconds per day will it lose in a place where the acceleration due to gravity is decreased by $\frac{1}{100}$ per cent ?

2. A simple pendulum of length 3 feet, oscillates through an angle of 12 degrees on either side of the vertical. Using the principle of conservation of energy, find the maximum velocity. Compare this with the maximum velocity obtained by considering the motion to be simple harmonic.

3. Two pendulums whose lengths are l and l_1 respectively begin to oscillate at the same instant, and after x oscillations they beat together again. If l, the greater, be known, find l_1.

4. At what height will a seconds pendulum beat 3588 times in an hour, the earth's radius being taken as 3958 miles.

5. If a seconds pendulum loses 10 seconds per day at the bottom of a mine, find the depth of the mine and the number of seconds that the pendulum would lose when half way down the mine.

6. A seconds pendulum hangs against a wall inclined at an angle θ to the horizontal. Show that the time for a small oscillation is

$$\pi\sqrt{\frac{l}{g \sin \theta}}.$$

7. A balloon ascends with a constant acceleration and reaches a height of 900 feet in one minute. Show that a pendulum clock, which has a seconds pendulum and is carried in the balloon, will gain at the rate of about 28 seconds per hour.

8. A cage lift is descending with unit acceleration ; show that a pendulum clock, which has a seconds pendulum and is carried in the lift will lose at the rate of about 56 seconds per hour.

Additional Solved Examples

Ex. 1. A planing machine is working with its stroke set at 4 feet, and is planing a piece of work 3 feet long, the tool clearing the work 6 inches at each end. There is a constant frictional resistance equivalent to 45 pounds at the tool, and the cutting resistance is 560 lbs. more. The machine makes 20 complete strokes per minute. Find its average rate of working, in horse-power.

If the cutting stroke takes twice as long as the return stroke, find the average speed of each, and if the highest cutting speed is $1\frac{1}{2}$ times the mean, find the greatest rate of working.

In its forward stroke, the tool moves a distance of 4 feet and cuts 3 feet length of wood. Hence the work done $=(45\times 4)+(560\times 3)=1860$ ft.-lbs.-wt. per forward stroke.

In its back stroke, the resistance is due to friction of the tool only. Hence the work done $=45\times 4=180$ ft.-lbs.-wt. Since the machine makes 20 complete strokes, it makes 10 forward and 10 return strokes per minute.

Hence the total work done per sec. $=\dfrac{1860\times 10}{60}+\dfrac{180\times 10}{60}$

$=340$ ft.-lbs.-wt.

$$\therefore \quad \text{H. P.}=\frac{340}{550}=0{\cdot}618.$$

Time taken for one cutting and one back stroke $=\frac{1}{10}$ min.

Hence the time for one cutting stroke $=\frac{2}{3}\times\frac{1}{10}$ min.

Distance described during this time $=4$ feet.

$$\therefore \quad \text{The average forward speed} = \frac{4\times 30}{2}$$

$$= 60 \text{ ft./minute.}$$

Similarly, the average return speed $=4\times 30$
$=120$ ft./minute.

Now maximum cutting speed $=\frac{3}{2}\times 60$ ft. per minute
$=\frac{3}{2}$ ft./sec.

$$\therefore \quad \text{Maximum H. P.} = \frac{(45+560)\times 3}{550\times 2} = 1{\cdot}65.$$

Ex. 2. In a ballistic pendulum for estimating the velocity of rifle bullets, a block of wood weighing 10 lbs. was suspended by two parallel strings as shown in the figure. When a bullet weighing 0·4 oz. was fired into the block in a horizontal direction, passing through the centre of gravity of the block, it was observed that the block rose 6·95 inches. Find the velocity of the bullet.

Let v ft./sec. be the initial velocity of the bullet, and V ft./sec. the velocity with which the block begins to move.

The total momentum of the bullet and the block in a horizontal direction must remain constant, since there is no resultant force in that direction.

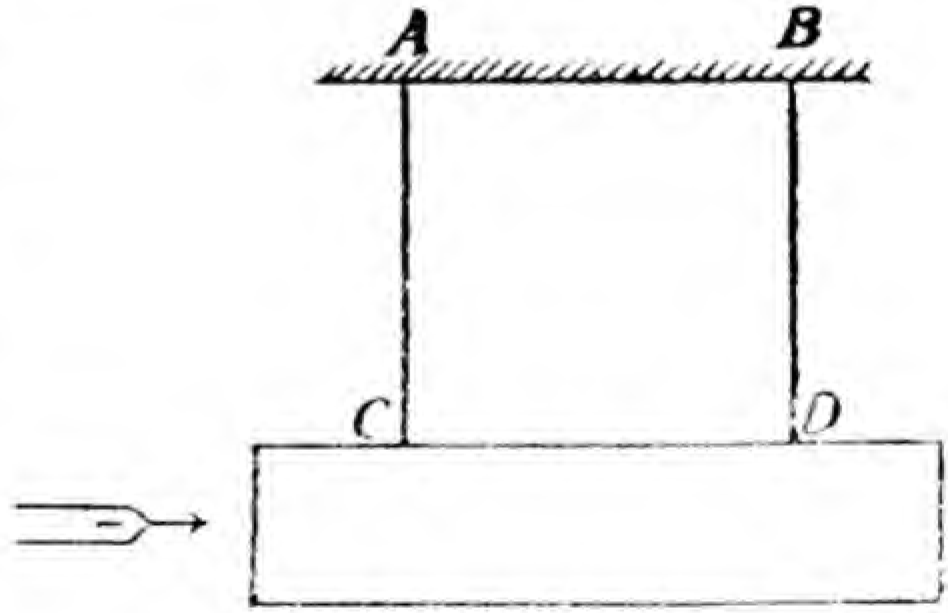


Fig. 100

$$\therefore \quad \frac{0{\cdot}4}{16}\,v = \left(10+\frac{0{\cdot}4}{16}\right)\text{V},$$

i.e. $\quad v = 10{\cdot}025\times 40 \;\; \text{V},$

$$v = 401 \text{ V} \qquad \ldots\ldots\ldots(1)$$

The block and the bullet now swing forward, and all their K. E. is changed to potential energy, when they reach their highest point.

$\therefore \quad \frac{1}{2}(10\cdot025)\ V^2 = 10\cdot025 \times 32 \times \dfrac{6\cdot95}{12}$ ....(2)

i.e. $\quad V^2 = \dfrac{64 \times 6\cdot95}{12}$,

or $\quad V = 6\cdot06$ feet per sec.

$\therefore \quad v = 401 \times 6\cdot06$
$= 2430\cdot06$ feet per sec.

Ex. 3. Prove that when two heavy particles are projected in the same vertical plane from two fixed points with equal velocities so that they collide, the sum of their angles of projection must be constant.

Let A and B be two fixed points, such that $AC = d$ and $BC = d_1$. Let the particles be projected from A and B with the same velocity u making angles α and θ with the horizontal respectively. Since they collide at P after a time t, we have,

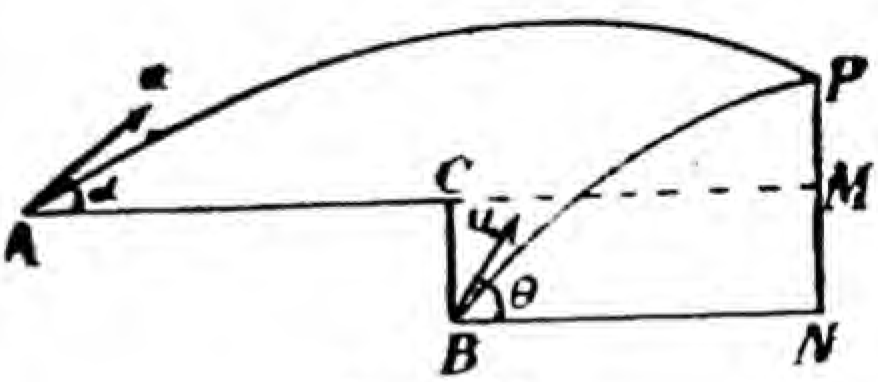


Fig. 101

$AM = u.\cos\alpha.\,t$

$PM = u.\sin\alpha.\,t - \frac{1}{2}gt^2$

$BN = u.\cos\theta.\,t$

$PN = u.\sin\theta.\,t - \frac{1}{2}gt^2$

Therefore, $AM - BN = AM - CM = d$

or $\quad ut(\cos\alpha - \cos\theta) = d$ ....(1)

Again, $\quad PN - PM = d_1$

$\quad ut(\sin\theta - \sin\alpha) = d_1$ ....(2)

Dividing (1) by (2), we have,

$$\frac{\cos\alpha - \cos\theta}{\sin\theta - \sin\alpha} = \frac{d}{d_1}$$

or $\tan\dfrac{\alpha+\theta}{2}=\dfrac{d}{d_1}=\text{constant}$

$\therefore$ $\alpha+\theta=\text{constant}.$

Ex. 4. Two masses M_1 and M_2 are connected by a fine inelastic string which passes over a smooth fixed pulley. The larger mass M_2 rests on a horizontal table ; M_1 is then raised h feet vertically above its position of rest and let fall. Show that the interval between M_2 leaving the table and returning to it again is $\dfrac{2M_1}{M_2-M_1}\sqrt{\dfrac{2h}{g}}$.

The velocity of M_1 when it has fallen h feet is $\sqrt{2gh}$. At this time the string becomes taut and the mass M_2 is jerked off the horizontal table.

The velocity imparted to M_2 is given by

$$(M_1+M_2)\,v=M_1\sqrt{2gh}$$

or $$v=\frac{M_1}{M_1+M_2}\sqrt{2gh} \qquad \ldots\ldots\ldots(1)$$

Now the common acceleration f of the two masses during subsequent motion is given by

$$M_1g-T=M_1f$$

$$T-M_2g=M_2f,$$ where T is the tension of the string.

$\therefore$ $$f=-\frac{M_2-M_1}{M_2+M_1}g \qquad \ldots\ldots\ldots(2)$$

The mass M_2 starts moving up with velocity v and acceleration f. Its velocity vanishes after time t given by,

$$O=\frac{M_1}{M_1+M_2}\sqrt{2gh}-\frac{M_2-M_1}{M_1+M_2}gt$$

$\therefore$ $$t=\frac{M_1}{M_2-M_1}\sqrt{\frac{2h}{g}}$$

After this time, since $M_2>M_1$, the mass M_2 will descend to the table. It will take $\dfrac{M_1}{M_2-M_1}\sqrt{\dfrac{2h}{g}}$ to come back to the table.

Hence the required time $= \dfrac{2M_1}{M_2 - M_1}\sqrt{\dfrac{2h}{g}}$.

Ex. 5. A uniform heavy chain hangs at rest over a smooth small peg; from one end of the chain one-fourth of its whole length is cut off; show that the pressure on the peg is instantaneously diminished by one-third.

Let mg be the weight of the chain. Originally the pressure on the peg is mg. When one-fourth of the length of the chain is cut off, the heavier portion begins to descend. If T poundals is the tension in the chain, we have

$$\frac{mg}{2} - T = \frac{m}{2}f$$

$$T - \frac{mg}{4} = \frac{m}{4}f$$

Solving these equations, we have,

$$T = \frac{mg}{3}.$$

Hence the pressure on the peg $= 2T$

$$= \frac{2mg}{3}$$

$\therefore$ Reduction in pressure $= mg - \dfrac{2mg}{3}$

$= \frac{1}{3}\,mg$ poundals.

Ex. 6. A stone is let fall from a height of 16 ft., and alights on a coiled spring which brings it to rest after passing over 1 ft.; the spring is so arranged that the force which it applies to the stone is constant and is equal to 20 lbs.-wt.; what is the mass of the stone?

The velocity acquired by the stone in falling through 16 feet from rest $= \sqrt{2 \times 32 \times 16} = 32$ ft./sec.

This velocity is reduced to zero, when the stone has moved 1 ft. on the spring. Therefore, if $(-f)$ is the acceleration in feet per sec.^{2} produced by the spring, we have,

$$O = 32^2 + 2\,(g - f)$$

or $$f = 32 \times 17 \text{ ft./sec.}^2$$

Therefore, if m is the mass of the stone in pounds,

$$mf = 20 \times g$$
$$m \times 32 \times 17 = 20 \times 32$$
$$m = 1\tfrac{3}{17} \text{ lbs.}$$

Ex. 7. If three bodies are projected simultaneously in the same vertical plane from the same point, prove that the area of the triangle formed by joining the three bodies at any instant will vary as the square of the time.

By Art. 40 (a), since g is the common acceleration of the three particles, their relative distances from the origin freely falling under gravity, are u_1t, u_2t, u_3t, after a time t ; u_1, u_2, u_3, being their initial velocities of projection.

$OA = u_1t$, $OB = u_2t$, $OC = u_3t$

Let, $\angle BOA = \alpha$, $\angle COB = \beta$.

$$\triangle ABC = \triangle AOB + \triangle BOC - \triangle AOC$$
$$= \tfrac{1}{2} u_1t . u_2t \sin \alpha + \tfrac{1}{2} u_2t . u_3t \sin \beta - \tfrac{1}{2} u_1t . u_3t \sin (\alpha + \beta)$$
$$= \tfrac{1}{2}t^2 [u_1u_2 \sin \alpha + u_2u_3 \sin \beta - u_1u_3 \sin (\alpha + \beta)]$$
$$= t^2 \times a \text{ constant quantity.}$$

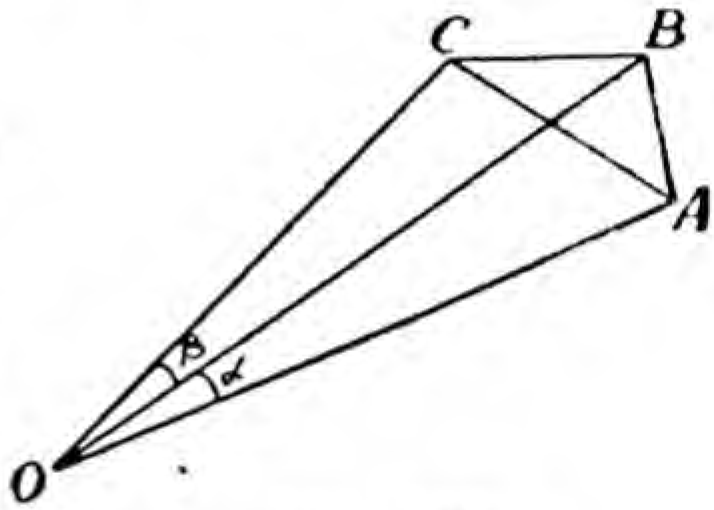


Fig. 102

Ex. 8. Two buckets, each weighing 16 ozs. are connected by a cord over a frictionless pulley, and an animal weighing 3 ozs. is placed in one of the buckets. The system is let go from rest, and at the end of $1\frac{1}{2}$ secs. the animal springs out of the bucket reaching a height of 8 inches above the starting point of the jump. Show that the velocity of the buckets while the animal is clear of them will be 5·11 ft. per second.

The bucket containing the animal moves down, and the common acceleration f is given by

$$\frac{19}{16} f = \frac{19}{16} g - T$$
$$f = T - g$$
$$\therefore \qquad f = \frac{3}{35} g \qquad \text{.........(1)}$$

The downward velocity of the animal in the bucket after $1\frac{1}{2}$ secs.

$$v=\frac{3}{35}\times 32\times\frac{3}{2}=\frac{144}{35}\text{ ft./sec.}$$

The velocity with which the animal springs up, so that it may travel up against gravity a distance 8 inches, is

$$O=v_1^2-\frac{2\times 32\times 8}{12}$$

or

$$v_1=8\sqrt{\frac{2}{3}}\text{ ft./sec.}$$

The jumping of the animal will disturb the velocity of the buckets, and if the new velocity of the buckets is v_2, we have by the law of conservation of momentum

Impulse on the bird = Impulse on the buckets.

$$\frac{3}{16}\left[8\sqrt{\frac{2}{3}}-\left(-\frac{144}{35}\right)\right]=\frac{32}{16}\left(v_2-\frac{144}{35}\right)$$

$$\therefore\qquad v_2=\frac{144}{35}+\frac{3}{32}\times 10{\cdot}64$$

$$=5{\cdot}11\text{ ft./sec.}$$

Ex. 9. A shell fired with velocity V at elevation θ hits an airship at height H, which is moving horizontally away from the gun with velocity v. Show that if

$$(2\text{ V}\cos\theta-v)(\text{V}^2\sin^2\theta-2g\text{ H})^{\frac{1}{2}}=v\text{V}\sin\theta,$$

the shell might also have hit the airship if the latter had remained stationary in the position it occupied when the gun was actually fired.

Referred to P as origin, PX as the axis of x and PY as the axis of y, the equation of the path of the shell is,

$$y=x\tan\theta-\frac{gx^2}{2\text{V}^2\cos^2\theta}\quad\cdots(1)$$

$$\text{AM}=\text{BN}=\text{H}.$$

The airship is moving along AB. The shell hits it at B. The

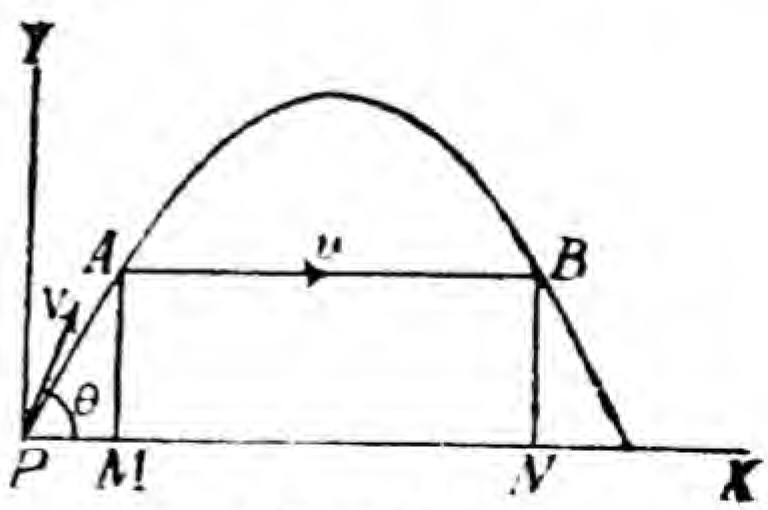


Fig. 103

shell might also have hit the airship if it had remained stationary at A when the gun was fired. Hence, if $PM=x$ and t the time which the shell takes in moving from P to B, we have $AB=vt$.

The co-ordinates of B are $(x+vt, H)$.

The co-ordinates of A are (x, H).

Substituting the co-ordinates of A and B in (1), we have,

$$H=x \tan \theta-\frac{g\, x^2}{2V^2 \cos^2 \theta} \qquad \text{.........(2)}$$

and

$$H=(x+vt) \tan \theta-\frac{g\,(x+vt)^2}{2V^2 \cos^2 \theta} \qquad \text{.........(3)}$$

Subtracting (2) from (3), we get

$$0=vt.\tan \theta-\frac{g\,(2x\, vt+v^2t^2)}{2V^2 \cos^2 \theta}$$

or $\quad 2V^2 \sin \theta \cos \theta=g\,(2x+vt) \qquad$ ....(4)

By considering the horizontal and the vertical distances travelled by the projectile in time t, we have,

$$V \cos \theta . t=x+vt \qquad \text{.........(5)}$$

$$H=V \sin \theta\, t-\tfrac{1}{2} gt^2 \qquad \text{.........(6)}$$

From (4) and (5), we get,

$$2V^2 \sin \theta \cos \theta=g\,(2V \cos \theta-v).\, t$$

or

$$t=\frac{2V^2 \sin \theta \cos \theta}{g\,(2V \cos \theta-v)} \qquad \text{........ (7)}$$

Substituting the value of t in (6) after a little simplification we have

$$(2V \cos \theta-v)\,(V^2 \sin^2 \theta-2gH)^{\frac{1}{2}}=v\, V \sin \theta.$$

Ex. 10. A shell of mass (m_1+m_2) is fired with a velocity whose horizontal and vertical components are u and v respectively ; and at the highest point of its path the shell explodes into two fragments m_1 and m_2. The explosion produces an additional kinetic energy E, and the fragments separate in a horizontal direction. Show that they strike the ground at

a distance equal to

$$\frac{v}{g}\left\{2E\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\right\}^{\frac{1}{2}}, \text{ between them.}$$

The time to reach the highest point $=\dfrac{v}{g}$ ......(1)

Let u_1 and u_2 be horizontal velocities of the fragments m_1 and m_2 after the explosion. Then by the law of conservation of momentum,

$$(m_1+m_2)u=m_1u_1+m_2u_2 \quad ......(2)$$

Also by the law of conservation ef energy,

$$\tfrac{1}{2}m_1u_1^2+\tfrac{1}{2}m_2u_2^2=\tfrac{1}{2}\,(m_1+m_2)u^2+E \quad ......(3)$$

From (2) and (3), we have

$$m_1u_1^2+m_2u_2^2=(m_1+m_2)\,\frac{(m_1u_1+m_2u_2)^2}{(m_1+m_2)^2}+2E$$

or

$$u_2-u_1=\left\{2E\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\right\}^{\frac{1}{2}} \quad ......(4)$$

Since the horizontal velocity of the shell remains constant, the fragment will take $\dfrac{v}{g}$ time to reach the ground. Hence the distance between their positions of striking the ground is

$$\frac{v}{g}(u_2-u_1)=\frac{v}{g}\left\{2E\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\right\}^{\frac{1}{2}}$$

Ex. 11. A body of mass m rests on a smooth table. Another mass M moving with velocity V collides with it. Both are perfectly elastic and smooth. The body m is driven at an angle θ to the previous line of motion of the body M. Show that its velocity is

$$\frac{2MV\cos\theta}{M+m}.$$

M strikes m at an angle θ with the line of centres. After impact, m moves along the line of centres with velocity v, say, and let M move with velocity V_1 at an angle ϕ, with the line of centres.

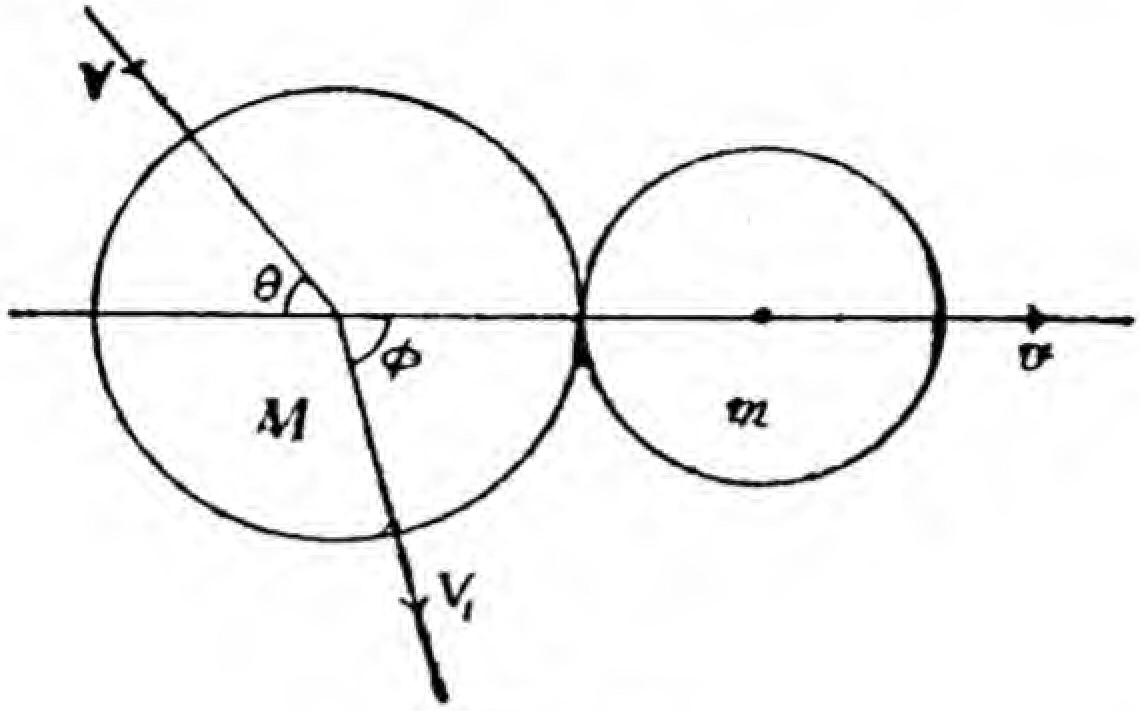


Fig. 104

Since on M, there is no force perpendicular to the line of centres, its velocity in that direction remains unchanged.

i.e., $$V_1 \sin\phi = V \sin\theta \qquad \dots (1)$$

The equation of momentum gives

$$MV_1 \cos\phi + mv = MV \cos\theta \qquad \dots (2)$$

Newton's Law gives

$$V_1 \cos\phi - v = -e\,(V\cos\theta - 0)$$
$$= -V\cos\theta \qquad \because e=1 \quad \dots (3)$$

From (3) and (2) we have

$$M\,(v - V\cos\theta) + mv = MV\cos\theta$$

or $$v = \frac{2MV\cos\theta}{M+m}.$$

Problems for Review

1. A projectile has initially a velocity whose horizontal and vertical components are u and v respectively. If R is the range and h the greatest height attained, show that

$$4\,\frac{h}{R} = \frac{v}{u} \text{ and } \frac{8h}{g} = \left(\frac{R}{u}\right)^2.$$

Also prove that the maximum horizontal range is $2h+\dfrac{R^2}{8h}$.

2. If v_1 and v_2 be the velocities at the ends of a focal chord of a projectile's path and u the horizontal component of the velocity, show that

$$\frac{1}{v_1^2}+\frac{1}{v_2^2}=\frac{1}{u^2}.$$

3. Show that it is possible to hit a mark y ft. height and distant x ft. horizontally from the thrower with a velocity u ft. per second provided

$$y+\sqrt{x^2+y^2}\leqslant\frac{u^2}{g}.$$

4. An inelastic sphere of mass M collides with another sphere of mass m. If the latter sphere was at rest before the collision, show that a portion $\dfrac{m}{M+m}$ of the kinetic energy is lost.

5. A gun of mass M fires a shell of mass m horizontally and the energy of the explosion is such as would be sufficient to project the shell vertically to a height h. Show that the velocity of recoil is

$$\left\{\frac{2m^2gh}{M(M+m)}\right\}^{\frac{1}{2}}.$$

6. A series of n elastic spheres whose masses are $1, e, e^2, e^3,$.........etc., are at rest, separated by intervals with their centres lying on a straight line. The first is made to impinge directly on the second with a velocity u. Show that finally the first $(n-1)$ spheres will be moving with the same velocity $(1-e)u$ and the last with the velocity u.

Prove that the final K. E. of the system is

$$\tfrac{1}{2}(1-e+e^n)u^2.$$

7. Two men, each of mass M, stand on two inelastic platforms each of mass m, hanging over a smooth pulley. One of the men, leaping from the ground, could raise his centre of gravity through a height h. Show that if he leaps

with the same energy from the platform, his C. G. will rise a height

$$\left(h - \frac{Mh}{2(M+m)} \right).$$

8. A train is going horizontally at 25 m. p. h. A rifle is pointed from it at a bird seen in a direction inclined at 60° to that of the train's motion, and a bullet is discharged. Supposing the bullet to move horizontally and uniformly until it kills the bird, which is flying uniformly and horizontally in a plane at right angles to the line of the train's motion, find the bird's speed, and its velocity relative to the train.

9. If a projectile whose mass is x lbs. is fired with a velocity u at a body whose mass is m lbs. advancing with a velocity v, the body will retain a velocity $\frac{mv - xu}{m+x}$, if the bullet is imbedded; but a velocity $v - \frac{x}{m}(u-u')$, if the bullet perforates and retains a velocity u'.

10. A particle is projected from a point in a horizontal plane with initial velocity v at angle of elevation α. Prove that its height above the plane after moving a horizontal distance x is $x \tan \alpha - \frac{x^2}{2R}(1+\tan^2 \alpha)$, where R is maximum range on the horizontal plane.

11. A steam hammer whose mass is 18 tons falls 9 feet, and remains in contact with a mass of iron for $\frac{1}{24}$ second; find (1) the average pressure exerted; (2) the compression effected.

12. Three elastic spheres of masses A, B, C are at rest on a straight line; A impinges with given velocity on B, and B subsequently on C; prove that A and B will have a second impact when

$$\frac{e^2+e+1}{e}$$

is greater than $\frac{B}{AC}(A+B+C)$, where e is the co-efficient of restitution.

13. A light string has masses m_1 and m_2 fastened to its ends. The middle part of the string lies stretched on a smooth horizontal rectangular table, and is perpendicular to two opposite edges of the table over which its ends hang. A particle of mass m is knotted on the part of the string which lies on the table. Show that the tensions of the two parts of the string are in the ratio $\frac{1}{m_2}+\frac{2}{m}:\frac{1}{m_1}+\frac{2}{m}$.

14. Corn falls uniformly through an opening to a floor at the depth h below; prove that the C. G. of the falling corn is at a distance $\frac{h}{3}$ from the opening.

15. A pendulum, whose length is l, makes m oscillations in 24 hours; when its length is slightly altered, it makes $(m+n)$ oscillations in 24 hours. Show that the diminution of length is $\frac{2nl}{m}$ nearly.

16. A second's pendulum loses 5 beats per hour (*i*) at the top of a mountain, (*ii*) at the bottom of a mine shaft. Assuming the earth to be a sphere of radius 4000 miles, determine (*i*) the height of the mountain and (*ii*) the depth of the mine shaft.

17. A smooth hollow sphere is revolving uniformly about a vertical diameter, at the rate of n revolutions per second. Show that a particle lying on the inner surface and rotating with it will rest either at the lowest point or at a depth below the centre equal to $\frac{g}{4\pi^2n^2}$.

18. Two equal masses, connected by an inextensible weightless thread which passes over a smooth fixed pulley, are hanging freely. Show that the tension of the thread is unaltered, when $\frac{1}{n}$ th of the mass is added to one and $\frac{1}{n+2}$ th of the mass is removed from the other.

19. A balloon ascends with a constant acceleration and reaches a height of 900 feet in one minute. Show that a

pendulum clock, which has a second's pendulum and is carried in the balloon will gain at the rate of about 28 seconds per hour.

20. A body is thrown up in a lift with a velocity u relative o the lift and the time of flight is found to be t. Show that the lift is moving up with an acceleration $\frac{2u - tg}{t}$.

21. A cannon ball has a range R on the horizontal plane. If h and h' are the greatest heights in the two paths for which this is possible, prove that $R = 4\sqrt{hh'}$.

22. Two particles of masses m and m', are moving in parallel straight lines at a distance a apart with given velocities v and v'; the particles are connected by a string of such a length that at the instant when it becomes taut it is inclined at an angle α to the two parallel straight lines; assuming that $v > v'$, show that the impulsive tension on the string at the instant it tightens is $\frac{mm'}{m+m'}$, $(v - v') \cos \alpha$.

23. A smooth ring, of mass M, is threaded on a string whose ends are then placed over two fixed smooth pulleys with masses m and m' tied on to them respectively, the various portions of the strings being vertical. The system being free to move, show that the ring will remain at rest if

$$\frac{4}{M} = \frac{1}{m} + \frac{1}{m'}.$$

24. A perfectly elastic ball is thrown from the foot of a plane inclined at an angle α to the horizon. If after striking the plane at a distance l from the point of projection it rebounds and retraces its former path, show that the velocity of projection is

$$\sqrt{\frac{gl(1 + 3\sin^2 \alpha)}{2 \sin \alpha}}.$$

25. A window is supported by two cords passing over pulleys in the frame-work of the window (which it loosely fits), the other ends of the cords being attached to two weights each equal to half the weight of the window. One cord breaks and the window descends with acceleration f. Show that

$f=g\ \dfrac{a-b\mu}{3a+b\mu}$, where μ is the co-efficient of friction, and a is the height and b the breadth of the window.

26. An engine pumps water through a hose, and the water leaves the hose with a velocity v; show that the rate at which the engine is working varies as v^3.

27. A cyclist and his machine together are of mass M lbs; if he ride, without pedalling, down an incline of 1 in m with a uniform speed of v ft. per sec., show that to go up an incline of 1 in n at the same rate he must work at the rate equal to

$$M\left(\frac{1}{m}+\frac{1}{n}\right)\frac{v}{550}\text{ H. P.}$$

28. Find the velocity acquired by a block of wood, of mass M lbs., which is free to recoil when it is struck by a bullet of mass m lbs., moving with a velocity v in a direction passing through its centre of gravity.

If the bullet be embedded a feet, show that the resistance of the wood to the bullet supposed uniform is $\dfrac{Mm}{M+m}\ \dfrac{v^2}{2ga}$ lbs. wt. and that the time of penetration is $\dfrac{2a}{v}$ secs. during which time the block will move $\dfrac{m}{M+m}\,a$ feet.

29. A ball of 2ozs. weight is projected with a velocity of 20 ft. per sec. from the lowest point of a circular tube, which is held in a vertical plane. If the radius of the circle into which the tube is bent is 4 ft., find at what height from the lowest point the ball passes from the outer to the inner face of the inside of the tube.

30. A weight of 6 lbs. is hung by strings 5 ft. long from two pegs 6 ft. apart. The weight is held with the strings in a horizontal plane and just taut, and is then let go. What will be the greatest pull on the pegs during the subsequent motion?

31. Two weights connected by a string, length a, are held in contact. One is allowed to fall, and after a time $t\left(<\sqrt{\dfrac{2a}{g}}\right)$

the other is let go. Show that after the second starts a time $\frac{2a-gt^2}{2gt}$ elapses before the string tightens, and that the K. E. lost then is $\frac{1}{2}\frac{w\,w'}{w+w'}gt^2$, where w and w' are the weights.

32. A body hangs from the roof of a carriage by equal strings attached to two other points, one directly in front of the other, so that the strings are at right angles. If the acceleration of the carriage is 4 ft./sec^2. in the direction of motion, find the ratio of the tensions of the strings.

33. A jet of water issues from a circular nozzle, whose diameter is $\frac{1}{2}''$, fixed at the end of a horizontal pipe, and strikes a rigid vertical plate placed at right angles to the direction of the jet. If 10 gallons of water per minute issue from the nozzle, find the horizontal force which would have to be placed at the back of the plate to keep it in its place, assuming that it entirely destroys the velocity of water.

34. If V is the velocity of a shot up the bore when leaving the muzzle of a gun, and u the velocity of the horizontal recoil of the gun, α the elevation of the gun, find the horizontal velocity of the shot when it leaves the gun, and prove that the angle β, which the line of departure of the shot makes with the horizontal, is given by $\tan\beta=\frac{v\sin\alpha}{v\cos\alpha-v}$, prove also that, $\tan(\beta-\alpha)=\frac{u\sin\alpha}{v-u\cos\alpha}$.

35. The hose of a fire engine makes an angle of 45° with the horizontal, and the jet strikes a wall at right angles' 50 ft. above the nozzle. Find the horizontal and vertical components of the velocity at the nozzle. The engine pumps 150 gallons of water per minute. Find the thrust exerted on the wall.

36. A particle is suspended by three equal strings of length a from three points forming an equilateral triangle of side $2b$ in a horizontal plane. If one string be cut, show that the tension of each of the other is instantaneously changed in the ratio

$$\frac{3a^2-4b^2}{2(a^2-b^2)}.$$

37. A smooth wire, bent into the form of a circle of radius r, is rotating about its vertical diameter with angular velocity ω. If a mass is strung on the wire, show that it will be in equilibrium when its angular distance θ, from the vertical diameter is such that

$$r\omega^2\cos\theta = g.$$

The mass is given a small displacement from its position of equilibrium ; show that it will oscillate about this position and that the period of oscillation is

$$\frac{2\pi}{\omega\sin\theta}.$$

38. A pendulum bob, weighing 8 ounces, rotates uniformly in a horizontal circle at the end of a light string 1 foot long, the other end of the string being fixed. If it makes 120 revolutions per minute, what is the tension in the string, and what angle does the string make with the vertical ?

At what angle and with what speed does the pendulum rotate if the tension in the string is equal to four times the weight of the bob ?

39. A motor car has a machine-gun mounted on it, and when the car is at rest the gun is fired horizontally and straight to the front for 15 seconds. Find the velocity of the car at the end of this time.

Rate of firing, 600 bullets per minute ; muzzle velocity of the bullets, 2400 feet/sec. ; weight of a bullet, $\frac{1}{2}$ oz ; weight of the loaded car, 18 cwt. ; resistance to motion of the car, 16 lbs.

40. A torpedo boat fitted with hydraulic propulsion took in 1 ton of water per second in a direction perpendicular to the direction of motion of the boat, and discharged it horizontally astern with a velocity of 37·25 feet per second, relative to the boat. With this discharge the steady speed of the boat was 12·6 knots. Find the resistance to motion of the boat. [One knot equals 6080 feet per hour].

41. The velocity of flow in a water main of 6 inches diameter is 5 feet per second. At one place the main is bent through an angle of 30°. Find the resultant force on the bend.

42. A light spring, whose inertia may be neglected, is such that a $\frac{1}{2}$ lb. weight will compress it 1 inch. It is compressed 2 inches and placed on a table so that it will expand in a vertical direction. If a $\frac{1}{4}$ lb. weight is put upon it and the spring is released, what is the velocity of the $\frac{1}{4}$ lb. at the instant it leaves the spring.

43. A boat weighing 500 lbs. is moving ahead under its own momentum at a speed of 5 ft./sec., when a man weighing 160 lbs. starts to run forward from the stern to the bow taking 2 seconds to run the 11 feet length of the boat.

(*a*) At what speed does the boat move while he is moving ?

(*b*) What is the speed of the man relative to water while he is running ?

(*c*) At what speed does the boat move after he stops ?

(*d*) How would the motion of the boat be affected if, instead of stopping at the bow, the man dived forward off the boat at the same speed ?

44. A mass M is moving on a smooth horizontal table with uniform velocity in a circle, being attached by an inextensible string to a fixed point. It hits another particle of mass m, lying at rest, which sticks to it. The tension of the string just before collision is T and just after collision is T′; show that $(m+\mathrm{M})\,\mathrm{T}'=\mathrm{MT}$.

45. A cannon ball of mass m is shot from a gun of mass M which is so mounted as to be free to recoil in a horizontal direction, so that its muzzle velocity, when fired horizontally, relative to the ground is V. Prove that the greatest range on a horizontal plane is $\frac{\mathrm{V}^2}{g}\left(1+\frac{m}{\mathrm{M}}\right)$, and that this is obtained by giving the gun the elevation 45°.

46. Two particles of masses m, m' lie on a smooth horizontal table connected by an inelastic string of length a ; m is projected in the plane at right angles to the string; prove that the initial radius of curvature of its path is $\frac{m+m'}{m}\,a$.

47. A perfectly elastic particle is let fall from a point P on the upper surface of a vertical hoop (centre O) and rebounds

from the inner surface of the hoop. Prove that after two rebounds it will be moving vertically if, $\tan\theta = 2\sin 4\theta$; where θ is the angle which OP makes with the vertical, and find the time which elapses before the particle arrives again at P.

48. A particle is projected with given velocity under the action of gravity from a point P so as to pass through a point Q ; show that if t_1, t_2 are the two possible times of flight

$$g\, t_1 t_2 = 2\ \mathrm{PQ}.$$

49. A uniform chain of length $2a$, hanging symmetrically in equilibrium over a thin smooth peg, is just displaced from its equilibrium position. Prove by the principle of conservation of energy that its velocity when just leaving the peg is $\sqrt{ag}$.

ANSWERS

Examples I

1. $\sqrt{3}$ ft./sec. ; 210° with the direction of 1 ft./sec. velocity.

2. 2 ft./sec. at 30° west of south.

3. $\sqrt{29}$ ft./sec. at an angle of elevation of $\tan^{-1}\frac{2}{5}$ with a horizontal line which is inclined at $\tan^{-1}\frac{4}{3}$ north of east.

5. $\dfrac{200}{1+\sqrt{3}}$; $\dfrac{100\sqrt{2}}{1+\sqrt{3}}$ **6.** 210 yds.

7. Perpendicularly to the stream ; $10\frac{5}{32}$ seconds.

Examples II

1. $449\frac{1}{3}$ ft. **2.** 1 minute. **3.** 330 ft.

4. $\dfrac{1280000}{3}$ ft./sec.^{2} ; $\dfrac{3}{3200}$ secs.

5. 848·5 ft./sec. **6.** 1 inch.

8. 10 secs. or 30 secs. **10.** $2\frac{1}{2}$ secs.

Examples III

1. $5\frac{5}{18}$ minutes. **2.** 15·31 minutes.

3. $8\sqrt{2}$ miles per hour. **4.** $5\frac{5}{11}$ secs.

5. $10\sqrt{13}$ m.p.h. at $\tan^{-1} 2\sqrt{3}$ with the direction of motion of the train moving 30 m.p.h.

6. $16\frac{4}{5}$ minutes ; $\frac{2}{5}$ miles. **9.** $v \sin \alpha$.

Examples IV

1. 400 ft. ; 10 secs. **2.** 32 ft./sec.

3. 900 ft. ; $7\frac{1}{2}$ secs. **4.** 400 ft.

6. 1 sec. ; 1·5 secs, **7.** 15 secs.

8. 96 feet.

9. $\frac{5}{4}$ secs. ; $\frac{15}{32}\, g$ ft. upwards from the point of projection.

10. $\frac{32}{93}$ ft./sec.^{2} ; 40 ft. per second.

11. (1) 800 ft. ; (2) 1300 ft. ; (3) 900 ft.

Examples V

1. $\sin^{-1}(\frac{3}{4})$; 9 ft. **2.** $65\frac{5}{48}$ ft.; $5\frac{5}{24}$ secs.

3. $\dfrac{1}{\sin a \cos \theta}\sqrt{\dfrac{2h}{g}}$.

5. The parts are 32, 96 and 160 feet; 3 secs.

7. $\sqrt{\dfrac{2}{g}}\,(\sqrt{b}-\sqrt{a})$, where b and a are the diameters of the larger and the smaller circles.

8. 45° with the vertical. **9.** 15°.

10. Distance up the hypotenuse = base.

12. Produce the line joining the lowest point of the given circle to the given point P to meet the circle at Q. Then, QP is the required line. [A circle can be drawn with P for the lowest point, touching the given circle at Q.]

13. $\sqrt{\dfrac{2}{g}}\,(\sqrt{b}+\sqrt{a})$, where b is the diameter of the upper circle and a that of the lower circle.

Miscellaneous Examples

1. 20 ft./sec^2. **6.** $6\frac{1}{4}$ ft.; $\frac{5}{8}$ sec.

7. 1200 ft./sec. **9.** 208 ft./sec.

10. $e^2\left(\dfrac{u^2}{2a}+h\right)$.

11. 1·83 ft./sec^2.; 2·69 ft./sec^2.; 57·8 sec.

12. 200 ft./sec^2.

13. 21 ft. from the last position; after $\frac{3}{5}$ sec.

14. $\frac{2}{3}$ sec. after the start of second particle; $14\frac{2}{3}$ ft. from O.

16. 60 m.p.h.; 44 secs.; 1936 ft.; 8 secs.

17. 80 ft./sec.

18. $3\frac{1}{2}$ sec. from the instant of throw of the first stone; 84 ft. from the point of projection.

19. $756\frac{1}{4}$ ft. **20.** 1 sec.; 0·7 sec.

23. $\frac{11}{15}$ ft./sec^2.; $\frac{22}{15}$ ft./sec^2.; 440 ft.

24. 9° 38′ S. of W; $29\frac{1}{5}$ mins.

25. 10 ft./sec^2.; 20 ft./sec^2.; 2, 5, 1 secs.

29. $\sqrt{gh}$; $\sqrt{gh}$ and 0, where h is the height of the plane.

33. 1120 ft./sec. **34.** 72 ft./sec. downwards.
35. $\frac{5}{19}$ ft. **36.** 256 ft.

37. $\frac{100}{981}$ cm., $\frac{50\sqrt{2}}{981}$ cm., $\frac{100\sqrt{3}}{2943}$ cm.

39. 13·8′, 655 yds.

Examples VI

3. 200 tons wt. ; 1433600 ft./sec. units.
4. 30 cms. ; 12 cms. **6.** 440 lbs. wt.
7. 144 lbs. **9.** 440 ft.
11. 141·4 cms. per sec. ; 165·1 gm. wt.
12. 17·28 ft.

Examples VII

1. (*i*) 4 ft./sec^2. ; (*ii*) $7\frac{7}{8}$ lbs. wt. ; (*iii*) 20 ft./sec. ; (*iv*) 50 ft. **2.** 1·68 secs. ; 5·04 ft.

3. 7·16 ft. ; 21·5 ft. **4.** $\frac{m}{2}$.

5. 4 ft. ; $3\frac{3}{4}$ secs. **7.** 4 ft.

Examples VIII

1. $\frac{11}{28}g$; ·3.

2. The larger mass descends with acceleration $\frac{2\sqrt{3}-3}{9}g$.

3. $16\sqrt{5}$ ft. per sec. ; 80 ft./sec.
4. 4·97 lbs. wt. **5.** 107 lbs. wt. (nearly).
6. 24·8 lbs. wt. ; $67\frac{1}{2}°$. **7.** $2\frac{1}{8}$ tons ; 1·6 tons.
9. 2·2 ft./secs^2. **10.** 3600 ft.
11. 16 secs. **12.** 3674 ft.
13. 2 min. 51 secs. ; 5640 ft.

Examples IX

1. 896000 ft. lbs./sec.
2. 10 ft. lbs./sec. ; (*i*) $\frac{5}{8}$ lb. wt. ; (*ii*) $3\frac{1}{8}$ lb. wt.
3. $11\frac{1}{28}$ lbs. wt. **4.** 4800 ft. lbs./sec.
5. $10\frac{5}{7}$ cwt. wt. ; $\frac{1}{480}$ second. **6.** 6·8 ft.

Examples X

1. 616 ft. lbs. per second. **2.** $21\frac{3}{5}$ H.P.

3. $0{\cdot}29$ ft./sec.^{2} **4.** $\frac{12375\ N}{85204\ n}$ tons.

5. $37\frac{29}{33}$ H.P.

6. $0{\cdot}146$ tons wt.; 220 ft. approximately.

Examples XI

1. $\frac{375}{18}$ poundals ; 4 secs. **2.** 8 ft./sec.
3. 1431 ft. per sec. nearly. **4.** 15 H.P.
5. $1{\cdot}07$ tons-wt. **6.** 2 secs. ; 8 ft,
7. 6 ft. lbs. ; 50 lbs. wt.
9. (-30) ft. in the original direction ; 75600 units of K.E. ; 103950 units of K.E.
10. 3.
11. (*a*) 280 units of momentum ; $\frac{1}{4}\times(560)^2$ units of K.E.
(*b*) 280 units of momentum ; $17\frac{1}{2}$ units of K.E. ; 35 poundals.

Examples XII

1. 1 : 3
3. 12 ft. away from the tower ; 6 ft. away horizontally and 16 ft. below the top.
4. $40\sqrt{6}$ ft./sec. ; $\alpha=45°$. **6.** $2h$; $2\sqrt{gh}$.
7. 15° or 75°.
8. $12\sqrt{29}$ ft./sec. at an angle $\tan^{-1}\frac{2}{5}$ with the horizontal.
10. $60\sqrt{3}$ ft. **11.** h ; $\sqrt{\frac{h}{g}}$ units of time.
12. The directrix is a horizontal line passing through the highest point of the circle. **13.** 1760 ft.

Examples XIII

1. $\frac{2uv\sin\theta}{g}$. **2.** 51° . 20′ ; 88°.
4. 11 ft. 11 in. **5.** $u\cos\alpha\sec\beta$ ft. per sec.
6. $\frac{u^2}{2g}$. **7.** $\tan^{-1}\left(\frac{1}{100}\right)$.

9. At a distance $a \left\{ \sqrt{\frac{H}{H-h}} \right\}$ from the pole.

Examples XIV

1. $\frac{1}{2^{\frac{1}{4}}}$.

2. $\frac{1}{2}$; 8 ft. poundals.

3. $4\frac{1}{50}$ ft. tons.

4. $\frac{1}{2}$.

6. 0·84 ft./sec. ; 3·696 ft./sec. ; 8·68 ft./sec.

Examples XV

1. $(1+e)$ 10 ft./sec.

6. The path of the ball is always parallel to a diagonal and it always returns to the point of projection.

7. $2eh \sin 2\alpha$ ft.

Examples XVI

1. 48 revolutions per minute.

2. 36·8 revolutions per minute.

3. Pull of 12·5 lbs. wt. at A and push of 12·5 lbs. wt. at B. Pull of 26·8 lbs. wt. at A and push of 1·7 lbs. wt. at B.

4. 2·06 secs. ; 5·77 lbs.

5. $23^{\circ}. 25'$; 1·63 lbs.

8. 153 *r.p.m.*

9. $40\sqrt{2}$ ft./sec.

10. $\tan\theta = 0{\cdot}112$; $\mu = 0{\cdot}112$.

12. 6·18 inches.

13. 50·4 tons wt. ; $9\frac{1}{2}$ inches ; 63 tons wt.

17. $\frac{10}{\pi^2}$ ft.

Examples XVIII

1 π ft./sec. ; $6\pi^2$ ft./sec.^{2} ; $\frac{\pi\sqrt{3}}{2}$ ft./sec. ; $3\pi^2$ ft./sec.^{2}

4. 3 inches ; 0·393 seconds.

5. 25 cms. nearly.

9. 35·08 poundals ; 28·92 poundals.

Examples XIX

1. 4·32 seconds. **2.** 2·048 ft./sec.; 2·052 ft./sec.

3. $l_1 = \left(\frac{x}{x+1}\right) . l.$ **4.** 13·19 miles.

5. 1630 yards nearly; 5 secs.

Problems for Review

8. $25\sqrt{\ }$ miles an hour; 50 m.p.h., 120° with the direction of train's motion.

11. (1) 766,080 lbs. wt. (2) 6 inches.

29. 5·5 ft. **30.** $11\frac{1}{4}$ lbs. wt. **32.** 9 : 7.

33. 1·02 lbs. wt. **35.** 44·2 lbs. wt.

38. $8\pi^2$ poundals; $\theta = \cos^{-1} \frac{2}{\pi^2}$; $\theta = \cos^{-1} \frac{1}{4}$; $8\sqrt{2}$ radians/sec.

39. 1·77 ft. per second.

40. $\frac{1}{2}$ ton wt.

41. 4·97 lbs. wt.

42. 3·26 ft. per second.

43. (*a*) 3·66...ft./sec. (*b*) 9·16 ft./sec. (*c*) 5 ft./sec. (*d*) 3·66...ft./sec.

Examples XIX

1. 4·32 seconds. **2.** 2·048 ft./sec. ; 2·052 ft./sec.

3. $l_1 = \left(\dfrac{x}{x+1}\right) . l.$ **4.** 13·19 miles.

5. 1630 yards nearly ; 5 secs.

Problems for Review

8. $25\sqrt{}$ miles an hour ; 50 m.p.h., 120° with the direction of train's motion.

11. (1) 766,080 lbs. wt. (2) 6 inches.

29. 5·5 ft. **30.** $11\frac{1}{4}$ lbs. wt. **32.** 9 : 7.

33. 1·02 lbs. wt. **35.** 44·2 lbs. wt.

38. $8\pi^2$ poundals ; $\theta = \cos^{-1} \dfrac{2}{\pi^2}$; $\theta = \cos^{-1} \frac{1}{4}$; $8\sqrt{2}$ radians/sec. **39.** 1·77 ft. per second.

40. $\frac{1}{2}$ ton wt. **41.** 4·97 lbs. wt.

42. 3·26 ft. per second.

43. (*a*) 3·66...ft./sec. (*b*) 9·16 ft./sec. (*c*) 5 ft./sec. (*d*) 3·66...ft./sec.